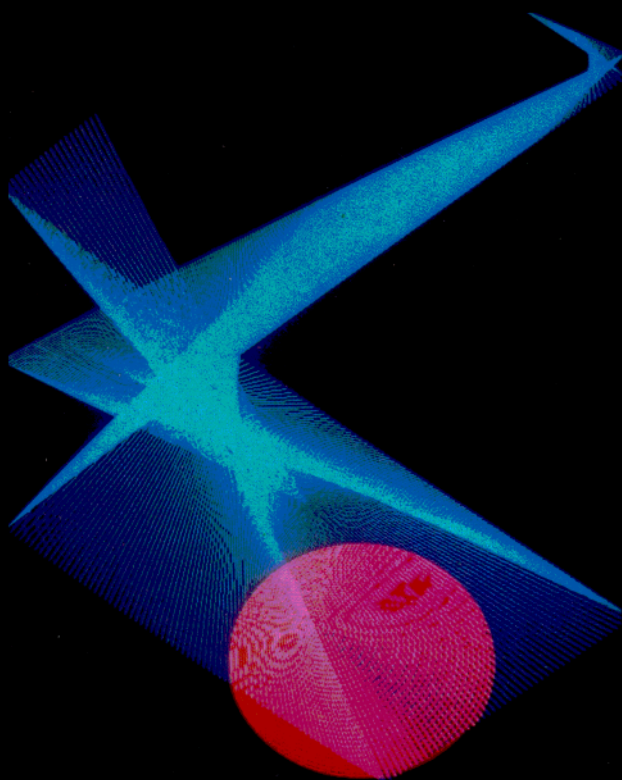


HOW AND WHY IN PHYSICS

1 KINEMATICS



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Between the formal physics textbooks, problem manuals and general expository books lies the domain of non-formal physics -- the world of content-specific strategies and styles of reasoning that practising physicists employ but do not fully articulate in print. This book tries to capture some elements of this world through the medium of a teacher-student dialogue. The dialogue runs through the many conceptual barriers that most critical students and teachers face in different topics of physics, and offers helpful points, clarifications and insights.

The book should be a valuable accompaniment to, but not a substitute for, the standard introductory text books of physics at senior secondary and introductory college level. It is especially suitable for teachers' orientation courses, and talent-nurture programmes for promising students at that level.

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PREFACE

The imagined student in this book is concurrently going through a standard introductory course in physics. The student is familiar with most parts of the course and has also practised solving problems. Yet she finds many points unclear and elusive, and she is keen on understanding them critically. The teacher respects the spirit of the student, argues with her on equal terms, and attempts to clarify the points and offer additional insights. The result is a dialogue through different topics of physics, sometimes dwelling on a minor confusion, other times discussing a major conceptual hurdle. Yet the teacher tries to explain with restraint, and judiciously leaves a point when further going might be too technical or advanced.

We believe readers who share the student's doubts on this or that point in introductory physics will find the dialogue useful, and gain some fresh clarifications and insights. But more important, after the book is over, they, like the insatiable student here, should be ready to ask more questions, and yet more questions

The work on this series got started mainly due to encouragement by Professors B. M. Udgaonkar and V. G. Kulkarni. Prof. K. Chandrasekhar (formerly of University of Madurai) offered useful comments on the manuscript. We record our gratitude to them. It is a pleasure to thank Shri. Anand Ghaisas for help concerning the artwork of the book.

" You are wrong, Goldmund. You assume that this zealous problem-solver continuously solves problems a teacher poses for him. But he can also ask himself questions; they can arise within him as compelling forces "

From *NARCISSUS AND GOLDMUND*
(Hermann Hesse)

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1

Units And Dimensions

Base Dimensions

S1 *I am sometimes puzzled by the phrase 'dimensions of a physical quantity'. Does the word 'dimensions' refer here to some kind of size ?*

T1 No. In ordinary language the word 'dimension' usually refers to size, and this is the sense in which it is often used in physics also, as in 'radiation of wavelength large compared to atomic dimensions'. However, in a phrase such as 'dimensions of angular momentum' the word has a very different meaning.

Dimensions of a physical quantity are related to the base units required to specify it; e.g. the unit of angular momentum is $\text{kg m}^2 \text{s}^{-1}$, and the dimensions of angular momentum are written as $[M][L]^2[T]^{-1}$. In the same way the unit of resistance in S.I. is $1 \Omega = 1 \text{ W A}^{-2} = 1 \text{ kg m}^2 \text{s}^{-3} \text{ A}^{-2}$ i.e. the dimensions of resistance are $[M][L]^2[T]^{-3}[A]^{-2}$.

S2 *We also talk of 'one-dimensional motion'; 'three-dimensional space' etc. What does the word 'dimension' mean here ?*

T2 The word 'dimension' in these usages means something entirely different from what I said in T1. Here it refers to the number of co-ordinates required to specify the position of a point, that is the number of variables required to span the given space (set of points). Thus motion along a line is one-dimensional because only one co-ordinate is enough to specify the location of a particle at an instant. Space is three-dimensional since we need three co-ordinates to specify the position of any point.

- S3 *I do not like this. The same word 'dimension' means different things in different contexts. We know this happens in ordinary language. But in science we should use a separate word for each distinct meaning, to avoid confusion.*
- T3 You are right. We indeed exercise this care in science mostly. Different technical terms in science have clear and distinct meanings. Occasionally, however, two or more distinct technical notions happen to be named by the same word. In such cases, the meaning should be clear from the context.
- S4 *From T1, dimensions of a physical quantity and its units seem to be much the same thing. Is there any difference ?*
- T4 The particular units of a quantity depend on the choice of standards for the base quantities of the system. For example, the units of energy are $\text{kg m}^2 \text{s}^{-2}$ in S.I., and $\text{g cm}^2 \text{s}^{-2}$ in the C.G.S. system. The dimensions of energy in each case are $[M][L]^2[T]^{-2}$. Thus 'dimensions' of a quantity specify the combination of the base quantities (length, mass, time, etc.) required to describe it without specifying the standards for the base quantities employed in the system. In this sense, 'dimensions' of a physical quantity is a more general notion than the 'units' employed to measure the quantity.
- S5 *Does that mean dimensions of a quantity are the same no matter what system of units is employed ?*
- T5 Not always. If two systems of units employ the same base quantities, but differ only in the standards used for those base quantities, then what you said is true. Thus in mechanics both S.I. and C.G.S. system have mass, length and time as base quantities. But the units of mass and length (i.e. the standards employed for their measurement) differ : kg and m in S.I., g and cm in C.G.S. system. The dimensions of a given mechanical quantity are, therefore, common to both S.I. and C.G.S. The units of that quantity may, however, differ. For example, the dimensions of force are $[M][L][T]^{-2}$ in both S.I. and C.G.S. system; but the units of force are $1 \text{ N} = 1 \text{ kg m s}^{-2}$ in S.I. and $1 \text{ dyne} = 1 \text{ g cm s}^{-2}$ in C.G.S. system.

But it is wrong to think that systems of units differ only in the choice of standards for base quantities. Two systems can differ in the choice of base quantities themselves.

This is so even for S.I. and C.G.S. when we go beyond mechanics and consider electromagnetic phenomena. For example, in S.I. current (or equivalently charge = current $\times$ time) is considered a base quantity while in the Gaussian C.G.S. system it is a derived quantity, not a base quantity. Consequently, the dimensions of capacitance, for instance, are $[Q]^2[M]^{-1}[L]^{-2}[T]^2$ in S.I. and $[L]$ in the Gaussian C.G.S. system. Here you see, therefore, an example of a quantity whose dimensions (and, of course, units) differ from one system to another.

S6 *Can we have a quantity which has units but no dimensions ?*

T6 It is possible in a somewhat artificial way. For example, if a clock loses 1 minute each day, we may characterize its rate of losing time as 1 minute/day. This quantity has units but no dimensions. But if we choose a common unit of time for the numerator and the denominator, we can write the same quantity as 1/1440 which has neither any units nor dimensions. As another example, the efficiency of an electric bulb may be expressed as so many lumens/watt; but it can also be expressed by a number (with no units or dimensions) which gives the fraction of input power that appears as luminous radiation.

A more familiar example is : angle. Angle is defined to be arc length divided by radius; it is, clearly, dimensionless. Still, it is customary to give it a unit called radian which is defined to be the angle for which the arc length equals radius.

S7 *In S.I., the five base units m, kg, s, K, and A do appear to be independent; but the other two units, namely mol and cd, seem to be derived units. Mole is a unit for amount of substance, but we already have a unit of mass (kg); a candela is a unit for luminous intensity i.e. luminous flux per unit solid angle. But luminous flux is analogous to radiant power, and we already have a derived unit of power (1 W = 1 kg m² s⁻³). What is the point in defining these redundant units ?*

T7 A mole does not represent a fixed amount of mass. A mole of C¹² means a mass of 12 g of C¹², whereas a mole of O₂ means a mass of about 32 g. What is common is the number of elementary entities; atoms in case of C¹² and molecules in case of O₂. In many situations, the number of atoms or molecules in a substance is more significant than the mass of the substance. For example, if the gas law is expressed in terms of the number of molecules: $PV = k NT$, k is a universal constant. If the same law is expressed in terms of mass as $PV = \alpha mT$, α is not a universal constant,

but depends on the molecular mass of the gas. Thus it is very convenient to define a unit (mole) which corresponds to a fixed number (Avogadro's number) of elementary entities.

A lumen is a unit of luminous flux. It is defined to be that quantity of light which equals the quantity of light of frequency 5.40×10^{14} Hz having a radiant power of $1/683$ W. Many light sources do not radiate isotropically. To characterize the amount of light emitted in a given direction one defines another quantity called luminous intensity which is luminous flux in that direction per unit solid angle. The unit of luminous intensity is one candela : $1 \text{ cd} = 1 \text{ lm sr}^{-1}$.¹ The reason why we need these different units is twofold. First, usually only a fraction of the radiant power of a source lies in the visible spectrum, and this fraction may differ from source to source. Second, brightness of an object depends not only on the 'visible' part of the radiant power but is also determined by the sensitivity of the eye to different portions of the visible spectrum. This means two sources having the same radiant power need not appear equally bright to our eyes. Consequently, to measure luminous flux (or luminous intensity in a particular direction) a separate unit is necessary.

S8 *In S.I. seven base units are defined : meter, kilogram, second, kelvin, ampere, candela and mole. Is there some significance to the number seven here ? Does it mean that in nature there are seven fundamental quantities and the rest are all derived from them ?*

T8 No doubt you have got this impression from the word 'base quantities'. But your idea is wrong.

The number of base quantities, which particular quantities to take as base quantities, and the standards for the base quantities are all a matter of choice. As already remarked, current is a base quantity in S.I. but not in the Gaussian C.G.S. Thus if we restrict to mechanics and electricity there are four base units (m, kg, s, A) in S.I. but only three base units (m, kg, s) in the Gaussian C.G.S. system.

The flexibility in the number of base dimensions is nicely illustrated when you consider the system of units called "natural units" which are employed in relativistic quantum physics. Here we take $\hbar = c = 1$ (dimensionless) and have only one base dimension

¹ Note: Luminous intensity is related to radiant power per unit solid angle. This is unlike the usual definition of intensity which is power per unit area.

in terms of which all the other physical dimensions are expressible.

- S9 *I can understand that certain constants like $\hbar$ and c can be assigned value 1 in some system of units, but how can you 'take' them to be dimensionless? $\hbar$ and c have definite dimensions: $[M][L]^2[T]^{-1}$ and $[L][T]^{-1}$ respectively. How do you do away with these dimensions?*
- T9 Of course you cannot do away with the dimensions of $\hbar$ and c if you continue to use mass, length and time as base dimensions. The idea is to relate the dimensions of mass, length and time to one another so that $\hbar$ and c become dimensionless.
- S10 *I do not understand that.*
- T10 From the relation speed = length / time, speed is dimensionless if the dimension of length is the same as that of time. Next, from $E = mc^2$, energy and mass have the same dimension, since c is dimensionless. Further from $E = \hbar\omega = h\nu$ energy, and hence mass, have the dimensions of inverse time (or inverse length). Thus we may consider only one base dimension say length $[L]$. Dimensions of all quantities can then be expressed in terms of this base dimension.
- In natural units:
Time : $[L]$, Mass : $[L]^{-1}$, Energy : $[L]^{-1}$, Momentum : $[L]^{-1}$.
- S11 *Do I take then that at least one physical base dimension is necessary, no matter what the system of units?*
- T11 Not at all. In relativistic quantum physics involving gravity we sometimes take $\hbar = c = G = 1$ (dimensionless) which implies that all the quantities in this system are dimensionless.¹
- S12 *All this sounds bewildering. What happens to the requirement that physical equations should be dimensionally consistent i.e. the powers of $[L]$, $[T]$, $[M]$ etc. should*

¹ The dimensions of G in terms of $[M]$, $[L]$ and $[T]$ are $[M]^{-1}[L]^3[T]^2$ and in natural units they are $[L]^2$. Thus if G is dimensionless so is length and hence all other quantities.

separately match on both sides of an equation ?

- T12 In a system that has $[L]$, $[T]$, $[M]$ as base dimensions, the powers of these three dimensions should match. In a system that has $[L]$, $[T]$, $[M]$, $[Q]$ as base dimensions, the powers of all the four dimensions must match. In natural units if we choose $[L]$ to be the base dimension, the powers of $[L]$ must match on either side of an equation.
- S13 *That still sounds very confusing. According to what you have said, an equation like $E = \frac{1}{2} mv^2$ is dimensionally wrong in S.I. or C.G.S. but dimensionally correct in natural units. Isn't there a contradiction ?*
- T13 There is no contradiction. Remember, a dimensionally wrong equation is necessarily wrong. But a dimensionally correct equation may be right or wrong actually. In other words, dimensional analysis can only rule out an equation but it can never establish any equation as actually correct. This power to rule out is naturally reduced if the number of base dimensions is reduced. Thus the equation $E = \frac{1}{2} mv^2$ can be ruled out on dimensional grounds in S.I. or C.G.S.; but in natural units you will need to invoke physics to rule it out.
- S14 *Let me turn to another point that often bothers me. Sometimes the constant of proportionality appearing in a physical law is simply taken to be unity. For example, Newton's second law is stated as $F \propto ma$ i.e. $F = k ma$ where k is a constant of proportionality, and k is then taken to be 1. What is the meaning of taking an unknown thing like k to be 1 ?*
- T14 It is this way. Look at the right side of the equation. The S.I. unit of mass is kg and that of acceleration is $m s^{-2}$. The unit of force on the left side is yet to be defined. If I take $k = 1$, the unit of force (newton) is one that produces an acceleration of $1 m s^{-2}$ in a mass of 1 kg. But I can as well take some different value of k , say $k = 5$. That will simply amount to having a unit of force which is different from 1 newton. Let us call this unit of force as 1 newt. Now $k = 5$ implies that the force required to produce an acceleration of $1 m s^{-2}$ in a mass of 1 kg is 5 newt i.e. 1 newton = 5 newt. In short, the value of k simply determines the size of the unit. Therefore, without any loss of generality, we can take $k = 1$ for convenience.

S15 *I encounter this phrase 'without any loss of generality' so often in physics. Please explain what this is supposed to mean.*

T15 Before that you must understand the words 'general' and 'special' as they are used in physics. If a statement or a law is true in all situations, we say it is true in general. If something is true in restricted or particular situations, we say that it is true in those 'special cases'. As an example, the general theory of relativity is applicable to all observers, while the special theory of relativity is applicable only to inertial observers. Another example that may be more familiar to you concerns collisions of two bodies. Here conservation of total momentum holds good in general, whereas total kinetic energy is conserved in the special case of elastic collisions.

Now, while handling a physical situation we may make some convenient choice without restricting or narrowing the problem to some particular situations. We then say that there is 'no loss of generality' in making such a choice. For example, consider the problem of motion of a charged particle in uniform electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. There is no loss of generality in aligning one of the coordinate axes, say the z-axis, along either $\mathbf{E}$ or $\mathbf{B}$. There is no loss of generality in taking the plane of $\mathbf{E}$ and $\mathbf{B}$ to be one of the x-y, y-z or z-x planes. But if we take both $\mathbf{E}$ and $\mathbf{B}$ to be aligned along the axes, say $\mathbf{E}$ along the z-axis and $\mathbf{B}$ along the y-axis, we shall lose generality because we shall be restricting the problem to the situations when $\mathbf{E}$ and $\mathbf{B}$ are mutually perpendicular.

S16 *Well, let us return to S14. It is clear to me now that we lose no generality in taking $k = 1$. But what about the dimensions of k ? When you write $1 \text{ N} = 1 \text{ kg m s}^{-2}$ you are obviously taking k to be dimensionless. Is a constant of proportionality always supposed to be dimensionless?*

T16 Not necessarily. Here we have chosen k to be dimensionless and defined the unit of force via the relation $F = k ma$. But we could have defined the unit of force in some other way. For example, we could have defined the unit of force via the Hooke's law $F = \kappa x$ (for some standard spring). Suppose we call this unit 1 hooke (in deference to Hooke who was Newton's rival!). The constant k in Newton's law $F = k ma$ will then have the units hooke $\text{kg}^{-1} \text{ m}^{-1} \text{ s}^2$.

As another example of a constant of proportionality being dimensional, remember Coulomb's law;

$$F = k \frac{q_1 q_2}{r^2}$$

In the Gaussian C.G.S. system, k is taken to be dimensionless, and thereby the dimensions of charge can be expressed in terms of $[M]$, $[L]$ and $[T]$; $[Q] = [M]^{1/2}[L]^{3/2}[T]^{-1}$. But in S.I. current (and hence charge) is defined independently via Ampere's law. Consequently in S.I., k is not dimensionless, but has the dimensions $[M][L]^3[T]^{-2}[Q]^{-2}$.

Significant Figures

- S17 *Let me move to another topic. The notion of significant figures is often discussed in connection with measurement. What is the meaning of significant figures in a measurement ?*
- T17 *This is best illustrated by an example. Suppose that the result of a length measurement is given to be 13.59 m. We say all the four figures here are significant if the length has been measured to an accuracy of 1 cm i.e. the actual length is established to lie between 13.585 m and 13.595 m.*
- S18 *But I can write the same number as 13.590 m. That makes the number of significant figures 5. Am I right ?*
- T18 *No. The zero in 13.590 is not significant.*
- S19 *Why not ?*
- T19 *Because, writing a zero in front of the number will not change the numerical value of the result but it changes the claim as to how accurate the measurement was. When you write the length to be 13.590 you are implying that the length has been measured to an accuracy of 1 mm i.e. the actual length lies between 13.5895 m and 13.5905 m. This is not the accuracy of the original measurement.*

S20 *That confuses me a little. I can always put any number of zeroes at the end of the decimal part of a number. How am I to know how many of these are significant or not?*

T20 Significant figures refer to the accuracy of measurement. When an experimentalist gives a result of measurement, he takes into account the accuracy of his procedure and rounds off the number he gives such that all the figures in the number are significant. Thus if he gives the length as 13.59 m, his measurement according to him is correct up to 4 significant figures, and I have no right to write it as 13.590 m or 13.5900 m etc. because that gives a misleading impression about the accuracy of the experiment. If, however, he quotes the result as 13.590 m, then it is understood that he means the last zero to be significant because he has measured the length to an accuracy of 1 mm. The number of significant figures is then five.

S21 *Does the number of significant figures depend on the units employed ?*

T21 No. For example, $13.59 \text{ m} = 0.01359 \text{ km} = 1359 \text{ cm} = 13590 \text{ mm}$. The number of significant figures is four in each case. The number of places of decimal differs. This is why it makes more sense to talk of accuracy of measurement in terms of the number of significant figures than saying that a certain thing is measured correct up to so many places of decimal.

S22 *Can you summarize for me the rules regarding counting of significant figures in a number ?*

T22 It is better to go by the simple meaning of significant figures as explained in T20 than to use any rules. Still, if you insist, you can see these rules from the following examples :

The case when there are no zeroes in the number is no problem. A length reported as 27.572 has 5 significant figures. A time interval given as 5.74 s has 3 significant figures.

A mass reported as 200.007 kg has 6, not 2, significant figures. The zeroes here are significant. One way of seeing this is we cannot get rid of them by multiplying the number by powers of ten.

$200.007 \text{ kg} = 200007 \text{ g}$; both have 6 significant figures.

But a mass written as 0.0007 kg has 1, not 4, significant figure. Here by multiplying by powers of ten we can get rid of the zeroes. $0.0007 \text{ kg} = 0.7 \text{ g}$ both have only 1 significant figure.

A measurement 1.51 m has 3 significant figures. If it is written as 1510 mm the last zero is not significant. However, if a measurement is reported as 1510 mm or 1.510 m or 151.0 cm, the last zero is significant.

To avoid confusion regarding the extra zeroes which appear when the same measurement is expressed in different units, it is conventional to write the measured number as a number with only one digit to the left of decimal point, multiplied by an appropriate power of ten. For counting the significant figures, then, the powers of ten are ignored. The remaining part is supposed to contain figures which are all significant.

Thus $200.007 \text{ kg} = 2.00007 \times 10^2 \text{ kg} = 2.00007 \times 10^5 \text{ g}$, all have 6 significant figures.

Similarly, $1.51 \text{ m} = 1.51 \times 10^{-3} \text{ km} = 1.51 \times 10^2 \text{ cm}$ have 3 significant figures.

If, however, a length is specified as $1.510 \times 10^{-6} \text{ m}$ then it is meant to have 4 significant figures and not 3.

S23 *What happens if a calculation uses quantities with different numbers of significant figures.*

T23 The answer should be rounded off to the lowest number of significant figures. For example, $5 \text{ kg} + 2.1628 \text{ kg} = 7.1628 \text{ kg} \approx 7 \text{ kg}$.

$$4.2 \text{ N} \times 1.532 \text{ m} \approx 6.4 \text{ J}$$

$$1525 \text{ m} / 94.3 \text{ s} \approx 16.2 \text{ m/s}$$

But remember, constants such as 2, π etc. appearing in a formula have no limit on their accuracy. Thus the constant 4 in the formula $A = 4 \pi R^2$ should not be thought of as having only 1 significant figure; strictly it has infinite number of significant figures (i.e. 4 is to be thought of as 4.00...). The rounding off of the final result, in this case, will be determined by the number of significant figures in the measured value of R. Also, the value of π should be written correct up to as many significant

figures as there are in R . For example, if $R = 1.55$ m then we should take $\pi = 3.14$ which gives $A = 30.2$ m². If $R = 1.5539$ m then we should take $\pi = 3.1416$ which gives $A = 30.343$ m².

Approximations

- S24 *Let us move to another thing that bothers me. We often make approximations in physics and say that something is negligible because it is small. But smallness and bigness of a quantity depends on the units employed. Then how do we justify such approximations?*
- T24 This matter is quite simple. 'Smallness' or 'bigness' is always in comparison to something else, even though this 'something else' is often not explicitly mentioned. In science when you talk of 'smallness' or 'bigness', be sure to consider a ratio of two similar quantities, i.e. quantities having the same dimensions. This ratio is dimensionless and its value will not change from one choice of units to another. 'Smallness' corresponds to this dimensionless ratio being much smaller than 1; 'bigness' corresponds to this ratio being much greater than 1.
- S25 *Can you illustrate this point ?*
- T25 In mechanics when we represent an extended body by a pointlike particle, we consider the dimensionless ratio of the size of the body, s , to the typical distance involved in the motion, d . If $s/d \ll 1$ then the particle approximation is good.¹ For example, a 100 m long train is approximately a point particle if we are dealing with its motion over hundreds of kilometers since the ratio s/d in this case is $\sim 10^{-3} \ll 1$. In the same way, the dimensionless ratio of interest for a simple pendulum is amplitude divided by the length of the pendulum. (This is the same thing as the angular amplitude.) The familiar formula $T = 2\pi\sqrt{l/g}$ is good in the approximation of this dimensionless ratio being small. Similarly Newtonian mechanics is valid in the approximation of the dimensionless ratio $v/c \ll 1$, where v is the speed of the body under consideration and c is the speed of light. We should try to cultivate the habit of looking for dimensionless ratios in a problem, especially when some approximations

¹ Often one is interested only in the translational motion of an extended object. It is then possible to represent the object by its centre of mass, no matter what its size is compared to the displacement.

are being made.

S26 You used words like 'this approximation is good under such and such conditions'. But 'good' and 'bad' are qualitative words. Do we not have a precise quantitative way of talking about approximations?

T26 Surely, there is. A quantitative statement regarding an approximation involves some dimensionless ratio. Let us call this dimensionless ratio α in some particular problem. Suppose some quantity in the problem, say A , depends on α . Mathematically it is usually possible to express $A(\alpha)$ as a power series in α , i.e.

$$A(\alpha) = A_0 + \alpha A_1 + \alpha^2 A_2 + \alpha^3 A_3 + \dots$$

For $\alpha < 1$ the successive terms in this series are expected to be smaller and smaller.[†] This brings out the important point that approximation is a matter of degree (or order as it is called in physics). In the crudest approximation, we can simply ignore all the terms except the first and write

$$A(\alpha) \approx A_0. \quad (\text{Zeroth order approximation})$$

By retaining more and more terms on the right side we can go on improving our approximation;

$$A(\alpha) \approx A_0 + \alpha A_1. \quad (\text{First order approximation})$$

$$A(\alpha) \approx A_0 + \alpha A_1 + \alpha^2 A_2. \quad (\text{Second order approximation})$$

$$A(\alpha) \approx A_0 + \alpha A_1 + \alpha^2 A_2 + \alpha^3 A_3. \quad (\text{Third order approximation})$$

And so on. Thus, if in a calculation involving α which is small, we consistently neglect powers of α higher than α^n , the results of the calculation can be claimed to be true up to n^{th} order of approximation in α .

S27 This seems too technical. Can you explain it with some examples?

[†] Strictly there could be problems of convergence of the series which we ignore here.

- T27 The discussion in T26 may seem a little technical, but I advise you to go over it carefully. The notion of order of approximation is very important in physics. If you are not clear about it, you will tend to look at the approximation procedures in physics as some kind of clever but adhoc tricks to simplify calculations and obtain 'desired' results. That is not so. Approximation is a well-defined quantitative procedure in science. As a first example consider the derivation of the result $f = R/2$ for a concave spherical mirror.

Box 1.1

Consider a spherical mirror with OM as its principal axis (See Fig. 1.1)

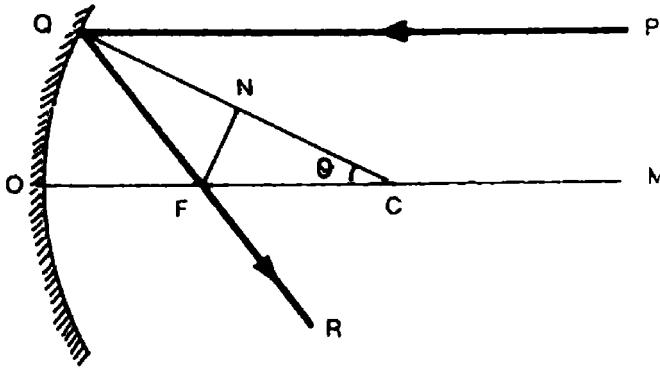


Fig.1.1 Locating the focus of a concave spherical mirror.

PQ is a ray incident along a direction parallel to the principal axis. QC is a radius of curvature of the mirror. The ray PQ is reflected along QR in accordance with the laws of reflection, $\angle PQC = \angle CQR$. QR intersects the principal axis at F. Since the incident ray PQ is parallel to the principal axis $\angle PQC = \angle QCF$. Thus $\triangle QCF$ is an isosceles triangle. FN is drawn perpendicular to QC; clearly $CN = R/2$. Let us denote distance FC by x , consequently $\cos \theta = R/2x$. The distance f between the pole O and F is given by

$$f = R - x = R \left(1 - \frac{1}{2 \cos \theta} \right) \quad (1)$$

This equation may be rewritten using the series expansion of the cosine function as

$$\begin{aligned}
 f &= R \left[1 - \frac{1}{2} \left(1 - \frac{\theta^2}{2!} + \dots \right)^{-1} \right] \\
 &= R \left[1 - \frac{1}{2} \left(1 + \frac{\theta^2}{2!} + \dots \right) \right]
 \end{aligned}$$

While writing the second step we have employed the binomial expansion.

The angle θ depends on how far the incident ray is from the principal axis; its value is maximum for rays incident near the rim of the mirror. If the aperture of the mirror is small compared to its radius of curvature, θ will be small for all the rays. We may then retain only the first term in the expansion of the cosine function to obtain an expression for f as

$$f^{(0)} = \frac{1}{2} R .$$

It is now apparent that this familiar result is only a zeroth order (in θ) approximation to the more general result given in Eq.(1). The superscript (0) on f indicates the order of approximation. A more accurate result, correct to second order in θ , is obtained if we retain one more term in the expansion,

$$f = \left(\frac{1}{2} - \frac{\theta^2}{4} \right) R .$$

Note that only in the zeroth order is f independent of θ . Thus only a beam of paraxial rays converges to a single point. The notions of focus and focal length, therefore, apply only to paraxial rays.

As another example, let us obtain an approximate expression for kinetic energy from the exact relativistic expression.

Box 1.2

The relativistic expression for the kinetic energy of a particle with mass m is

$$K = \frac{mc^2}{\sqrt{1 - \left(\frac{v}{c}\right)^2}} - mc^2 .$$

The first term is often referred to as the total energy to remind us that it includes kinetic energy and the so-called rest energy. When the rest energy is subtracted from the total energy we obtain the kinetic energy. The square root in the denominator of the first term can be expanded using binomial theorem to obtain

$$K = mc^2 \left[1 + \frac{1}{2} \left(\frac{v}{c} \right)^2 + \frac{3}{8} \left(\frac{v}{c} \right)^4 + \dots \right] = mc^2$$

As is evident, the expansion is in terms of the dimensionless ratio v/c . Now order by order approximations can be made; the lowest order approximation gives us the familiar non-relativistic expression for kinetic energy

$$K = \frac{1}{2} mv^2$$

Approximate expressions for kinetic energy with more and more accuracy can be obtained by retaining terms in the above expansion up to higher and higher powers of v/c .

In these examples we started with the exact expression and reduced it to different orders of approximation. Sometimes it is not possible to obtain the exact result at all, but approximate result may be obtained by neglecting terms involving the dimensionless ratio α relevant to the problem. In such a case the final result can be claimed to be correct up to n^{th} order in α provided the terms we have neglected contain powers of α greater than or equal to $(n+1)$.

Dimensional Analysis

S28 *Another point. I have read this somewhere that sine and cosine functions in any physical expression must have dimensionless arguments. Is that because sine and cosine are functions of angle θ and angle is always dimensionless?*

T28 That is true. But I can make a more general statement. Any trigonometric, logarithmic, exponential, gaussian, hyperbolic functions appearing in physics must have dimensionless arguments.

S29 *How do you see that?*

T29 It will help if you know one piece of mathematics: the series expansions of these functions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$\sinh x$ and $\cosh x$ being built out of e^x and e^{-x} also have their own series. Do not worry if you are not very familiar with these series. All you need to know is that they involve sums of different powers of the argument x . Now if x has some dimensions, adding to it x^2 or x^3 etc. will not make sense dimensionally. (You cannot add m to m^2 or m^3 .) Thus in order for these expansions to make sense, the only possible thing is that x be dimensionless.

S30 *So if an expression in whatever context in physics has a sine, cosine, log, exponential, hyperbolic or gaussian etc., I can be sure that the argument of these functions has got to be a dimensionless quantity. This is a very interesting point, but is it of any use?*

T30 This is no additional constraint to be imposed from outside. If you do your derivations properly, you will automatically get dimensionless arguments in these functions. You should use this rule as a check on your expressions. Thus if you ever get an expression containing $\sin(t/\omega)$ where ω is angular frequency, you can be sure it is wrong because t/ω is not dimensionless. You must always get the dimensionless product ωt (or its multiple) as argument in these functions. In the same way if in a correct physical expression, you get something like $\exp\{-x^2/a^2\}$ you can be sure that a has the same dimensions as x , because only then is the argument x^2/a^2 dimensionless.

S31 *That reminds me of another little doubt. When I look at an expression or equation in physics, should I take all the numbers such as 2, 1/3, π etc. dimensionless?*

T31 Depends. If the expression you are looking at is general in the sense no numbers have yet been put in for its parameters and variables, then any numerical figure that you see is dimensionless. For example, in the expression for kinetic energy, $K = \frac{1}{2}mv^2$,

the factor $\frac{1}{2}$ is dimensionless. You may, however, put $m = 10 \text{ kg}$ in this expression for calculating the kinetic energy of a particular object. The expression then becomes $K = 5 v^2$. The factor of 5 in this expression is not dimensionless.

As another example, consider the expression for the displacement of a particle in simple harmonic motion :

$$y = A \cos \left(\frac{2\pi t}{T} + \phi \right) .$$

2π in this expression is dimensionless. Now if you put numerical values for A , ω and ϕ in some particular problem, say $A = 2 \text{ cm}$, $T = 10 \text{ s}$, $\phi = \pi/4$, we get

$$y = 2 \cos \left(\frac{\pi t}{5} + \frac{\pi}{4} \right) .$$

In this equation you must note that 2 has the same dimension as that of y i.e. $[L]$; the number $\pi/5$ has the dimension of $[T]^{-1}$ and the number $\pi/4$ is dimensionless.

S32 *Could you summarize for me the use of dimensional analysis in physics ?*

T32 For physicists, dimensional analysis is second nature, so to speak -- it is used so much that one is not even conscious of it. First, it provides a ready check on the equations. A result like

$$v = \sqrt{T \mu} ,$$

(where v is the speed of transverse waves on a wire of linear density μ under tension T) can be discarded without going through its 'derivation' because the dimensions of the two sides do not match. (The correct formula is $v = \sqrt{T/\mu}$.) Similarly if in an expression we see factors like $\sin(\omega x)$, where ω is an angular frequency and x is distance, we can be sure that an error has crept in. Of course, dimensional analysis can never tell you that an equation is correct; but it can point out that a certain equation is definitely wrong.

Second, it helps you recall formulas and equations in physics. For example, when I try to write down the transformation equation for time in special relativity, I often linger over the second term in the numerator :

$$t' = \frac{t - ?}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

I do know the term contains x . Also, I know that it should go to zero as v/c goes to zero. Now I start looking at dimensions. The denominator has 1 and v/c , both dimensionless. That is fine. The term I am looking for has dimension of time. Aha! it is probably xv/c^2 . Yes, that checks with my memory too.

In the same way, if I have forgotten the units of some parameter, I just have to look at an equation containing that parameter to get it back. For example, the units of the gas constant R can be obtained by looking at the gas law equation $PV = nRT$. Since pressure is force divided by area, PV has units of energy. R , therefore, must have the units $J \text{ mol}^{-1} \text{ K}^{-1}$.

S33 *These seem to me rather trivial uses, though I agree they could be important in practice. Is that all to dimensional analysis ?*

T33 No. Often it goes beyond mere checking dimensional consistency or deciphering dimensions of quantities in an equation. For example, if the derivation of a result in physics is long and involved, it may help us to see if the important features of the result can be obtained without going through the derivation, by invoking dimensional arguments only. An example quoted in many books is the formula for viscous force F on a sphere of radius r moving with speed v in a medium of viscosity η . We start by writing

$$F = (r)^\alpha (v)^\beta (\eta)^\gamma.$$

Now if you go back to the dimensions of η , you can easily see that its dimensions are $[M][L]^{-1}[T]^{-1}$. Writing the dimensions of each quantity in the equation, we get

$$[M]^1[L]^1[T]^{-2} \sim [M]^\alpha [L]^{(\alpha+\beta-\gamma)} [T]^{-(\beta+\gamma)}$$

Finally, equating the powers of $[M]$, $[L]$ and $[T]$ we obtain

$$\alpha = 1, \quad \beta = 1, \quad \gamma = 1.$$

Therefore

$$F \sim \eta r v .$$

The proper derivation gives

$$F = 6\pi \eta r v .$$

Of course, you can never obtain a dimensionless constant like 6π from dimensional analysis. But it is nice to see that the dependence on the variables that you have in the correct formula has been predicted on the basis of dimensional considerations alone.

S34 *I have a doubt here. How do you know one should take only r , v , η on the right side above ? Why not take density ρ of the sphere, its acceleration a etc. ?*

T34 That is a very pertinent question. It is a basic limitation of this game of trying to 'derive' formulas from dimensional analysis that you can never logically arrive at what the relevant variables are in a problem. To guess the relevant variables, you must use your intuition, experimental information or whatever. So what we have done above is not really any alternative derivation of the viscous force formula. Rather, given that r , v , η determine F , we see that dimensional considerations alone are sufficient to determine the nature of dependence of F on these variables.

But in my view the real power of dimensional analysis comes into play when we go to a new, unfinished part of physics.

S35 *I do not quite appreciate that. Please elaborate.*

T35 Well, in an unknown domain, correct equations of physics cannot be 'served' to you. You need to discover them. Now, discovery is often a matter of playing a game - trying out ideas and hunches which may not be quite clear to the 'player' himself. Dimensional analysis is a real handy toy for this game.

S36 *That sounds very interesting. Any examples ?*

T36 I shall give you two examples. First, let us imagine what Niels Bohr might have been

thinking in early 1912 when he was at Manchester. Now you may have learnt that Bohr wanted to explain Balmer series in hydrogen spectrum, and eventually his model did explain it. That, no doubt, is true. But you must know that Bohr came to learn of Balmer's formula only in early 1913. The atomic problem was worrying him much earlier.

S37 *What was worrying Bohr about the atomic problem ?*

T37 The stability of the Rutherford atom was one problem. But Bohr also wondered as to what determined the size of an atom. You see there was already evidence from many sources that atoms are not point objects but have size of the order of 10^{-8} cm.

S38 *So what if the size is 10^{-8} cm ?*

T38 There is no way in Rutherford model to get this size out. Classically, an electron can revolve around a nucleus at any distance, small or big. Bohr was puzzled by the fact that there seemed to be a minimum to the orbital size.

S39 *What has that to do with dimensional analysis ?*

T39 A lot. Remember Bohr was exploring an unknown domain. He must have 'played' with various ideas to explain this size. And perhaps one of the ideas was based on dimensional analysis.*

S40 *I am thrilled. Please continue.*

T40 Anybody's first guess would be that the constants e (electronic charge) and m_e (mass of an electron) would appear in the eventual formula for atomic size. (A little reflection shows that the nuclear mass, being much greater than m_e , will at most provide small correction to the atomic size. Nuclear charge Z , however, is important. In what follows we take $Z = 1$ (hydrogen atom)). You cannot, however, get a quantity

* The actual historical process leading up to the discovery by Bohr of the atomic model was far more involved than has been suggested in the following paras. Our interest here, however, is not to describe the history of this discovery but to illustrate the exploratory use of dimensional analysis.

of the dimension of length from these two constants. So you should look for yet another constant. Now the speed of light c was, by then, known to be a fundamental constant of nature. Using e , m_e and c it is easy to see that the quantity $e^2/m_e c^2$ does have the dimension of length. Still that does not solve the problem. (Note : e^2 should be replaced by $e^2/4\pi\epsilon_0$ in S.I.)

S41 *Why not ?*

T41 When you substitute the values of e , m_e and c you find $e^2/m_e c^2$ to be 2.8×10^{-15} m which is far too small compared to the scale of atomic size. No, this quantity is not what we are looking for. On second thoughts, it is perhaps not surprising that the attempt above has not worked. The constant c has a role to play in relativistic domain. That is not the typical domain of electronic motion in atoms. We know that electronic energies in atoms are of the order of eV to keV. The rest mass of electron is about 0.5 MeV. This suggests that atomic size should be mainly determined by non-relativistic considerations. So discard the constant c from the game.

S42 *Well, then we are back to e and m_e . What might be Bohr's thoughts at this stage ?*

T42 Bohr obviously needed another constant. He was aware of the Planck's constant h which determined the minimum of energy (for a given frequency) that can be exchanged between matter and radiation. By a marvellous intuition, he guessed that the minimum in orbital size of an electron in an atom might not be completely unrelated to this, i.e. h might be a relevant constant for atoms too.

S43 *That is just a guess. What could be a clue to this guess, apart from intuition ?*

T43 The clue might have come from dimensional analysis. Try to construct a quantity with the dimension of length using e , m_e and h . You will discover that the combination $h^2/m_e e^2$ has the dimension of length. What is more, putting in the values of e , m_e , and h we get the length to be 2×10^{-10} m which is close to the order of atomic size. (Actually if you put $\hbar$ in place of h you get 0.53×10^{-10} m - almost exactly the size of the hydrogen atom!)

S44 *That looks like mere manipulation with constants. It is hardly a theory of atomic size.*

T44 You are right. At this stage Bohr was far from any theory. Yet he got the first important idea, namely that Planck's constant h was likely to appear in any theory of the atom. The rest is history.

S45 *What is your second example of the use of dimensional analysis in exploring unknown domains of physics ?*

T45 This relates to a curious, so far unproved, conjecture of the great physicist P. A. M. Dirac. Now while in the first example we have only speculated about Bohr's thinking, Dirac, we know, actually played the game of dimensional analysis with fundamental constants of nature.

S46 *What is Dirac's conjecture ?*

T46 Dirac obtained two dimensionless numbers from these constants, namely

$$\frac{e^2}{4\pi\epsilon_0 G m_e m_p} \quad \text{and} \quad \frac{4\pi\epsilon_0 m_e c^2 cT}{e^2} .$$

The first is the ratio of the electrostatic force

and the gravitational force between an electron and a proton. The second number represents the ratio of the radius of the universe (the present age of the universe T , which is estimated to be about ten billion years, multiplied by c , the velocity of light) and the so-called classical radius of the electron. When numerical values of the constants are substituted in these expressions the numbers turn out to be very large ($\sim 10^{40}$) and nearly equal (within an order of magnitude). Dirac was struck by the near coincidence of these two large numbers. He hypothesized that in some future theory they should turn out to be equal :

$$\frac{e^2}{4\pi\epsilon_0 G m_e m_p} = \frac{4\pi\epsilon_0 m_e c^2 cT}{e^2} .$$

S47 *That seems very speculative. In any case what is the implication of this equality ?*

T47 T in the equation above, recall, stands for the age of the universe. So, obviously, it is not constant in time. T increases with the age of the universe. Consequently, if Dirac's conjecture is true, one or more of the other 'constants' in the equality must

change with time. People have conjectured that the gravitational constant G might have (extremely weak) dependence on time. So far the conjecture remains unproved.

S48 *What is the moral of the story ?*

T48 Try to make dimensional analysis part of your instinctive response to any problem in physics. For any physical situation, look at the relevant quantities, their dimensions and orders of magnitudes; note any significant dimensionless ratios and understand the approximations involved in terms of these ratios. And, of course, do check the dimensional consistency of your equations. When the 'territory' is unknown, use dimensional analysis as one of the tools to make progress.

S49 *I am already a convert. Shall we move on ?*

2

Scalars And Vectors

Vectors add differently from scalars

- S1 I know the distinction between scalars and vectors : a scalar is a quantity that has only magnitude but no direction; a vector is a quantity that has magnitude as well as direction. What is so important about this distinction ?

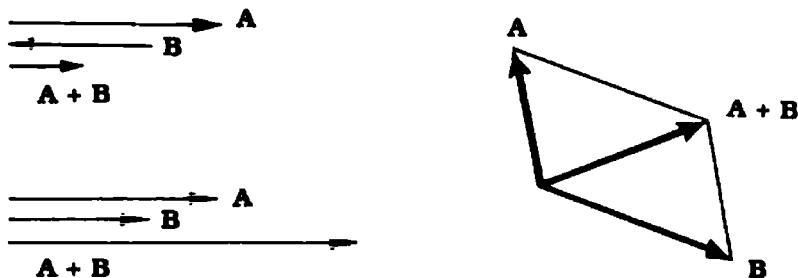


Fig.2.1 Parallelogram law of addition of vectors A and B leads to resultants of different magnitudes and directions, depending on the angle between them.

- T1 One important distinction is that vectors 'add' differently from scalars. Scalars add according to the usual rules of arithmetic, whereas vectors add according to the parallelogram law. This makes all the difference in so many situations in which we have to 'add' up physical quantities to obtain the resultant. For example, a body

which has received energy of 12 J from one source and 9 J from another has received a total of 21 J of energy. But a body subject to two forces of 12 N and 9 N may have a resultant force anywhere between 3 N and 21 N, depending on the angle between the two forces. (See Fig.2.1)

S2 *A scalar has magnitude only. Does it mean a scalar can never take negative values ?*

T2 No, there is no such restriction in general. Scalars like mass, speed and kinetic energy, are, of course, non-negative. But a quantity like potential energy is also a scalar and it can take any value, positive or negative. A scalar should not be associated with any direction in space, that is all. An electric charge is also a scalar because it is not associated with any direction in space. It can take any value, positive, negative or zero.

S3 *But sometimes positiveness or negativeness of a quantity itself can denote direction. For example, on a straight road velocity along one direction may be taken to be positive, and, in the opposite direction, negative.*

T3 That is a correct observation. The sign of a physical quantity may or may not have to do with direction. It depends on what quantity we are talking about. The sign of charge or potential energy does not have anything to do with direction. On the other hand the sign of displacement, velocity or force in one dimension is related to the direction of these quantities.

S4 *Why do you say that about one-dimensional vectors only ?*

T4 Because the directions of vector quantities in two dimensions (vectors in a plane) or of vectors in three-dimensional space cannot be characterized by merely assigning a sign to the numbers that denote their magnitudes. In one dimension (vectors confined to a line), however, there are only two possibilities of direction which can be taken care of by the algebraic sign + or -.

S5 *Does that mean there is no difference between scalars and one-dimensional vectors ?*

T5 None mathematically. Algebra of one-dimensional vectors (which means the rules of

operations on these quantities) is the same as that of scalars. For example, addition of two charges -5 C and $+2\text{ C}$ is -3 C . In the same way addition of two forces -5 N and $+2\text{ N}$ along the same line gives -3 N . The negative sign in -3 C has no relation to any direction. The negative sign in -3 N , however, gives the direction of the force on the given line.

S6 *Are there situations where one-dimensional vectors are relevant ?*

T6 Consider a ball thrown vertically upwards. Here velocity, acceleration, displacement and force are all one-dimensional vectors.

S7 *In what way is the algebra of vectors in two and three dimensions different from the algebra of scalars ?*

T7 We have seen that already in T1. But I repeat the point because it is very crucial. Scalars add like numbers in arithmetic. Vectors add according to the parallelogram law. If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then C , i.e. the magnitude of $\mathbf{C}$, is not equal to $A + B$ but is given by

$$C = \sqrt{A^2 + B^2 + 2AB \cos \theta} \quad ,$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. Only for $\theta = 0$, $C = A + B$.

Another way of seeing the difference between scalars and vectors is as follows. A scalar is associated with one real number, a vector in a plane with an ordered pair of numbers. In the same way, a vector in three dimensions can be associated with an ordered triplet of numbers. To add vectors, all one does is arithmetically add the corresponding numbers in the pairs or the triplets. For example, if vector $\mathbf{A}$ is associated with a pair $(3, -1)$ and vector $\mathbf{B}$ with the pair $(5, 3)$; then the pair associated with $\mathbf{A} + \mathbf{B}$ is $(8, 2)$.

S8 *These seem to be ordinary rules of addition. What is the difference ?*

T8 That is a serious error. The rule regarding addition of pairs (or triplets) given above does not follow from the arithmetic of numbers. It follows from the parallelogram law of vector addition.

Box 2.1

Consider, for simplicity, vectors in two dimensions. The resultant C of vectors A and B is

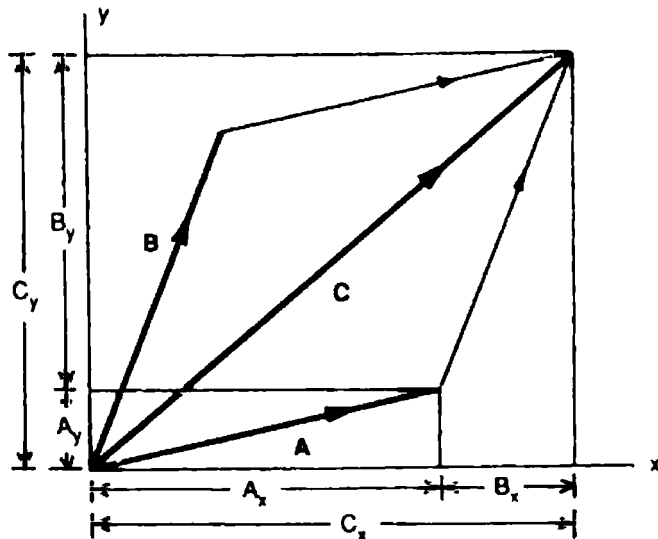


Fig.2.2 Vector addition rule in terms of components.

given by the parallelogram law as shown in Fig 2.2. We choose rectangular cartesian axes as shown in the figure. The components of the three vectors, along these axes, are (A_x, A_y) , (B_x, B_y) and (C_x, C_y) . It is quite evident from the figure that $C_x = A_x + B_x$ and $C_y = A_y + B_y$. Thus the law of addition of pairs follows from the parallelogram law. Its generalization to three dimensions is straightforward.

S9 *Is time a scalar or a vector ?*

T9 Of course, a scalar.

S10 *Why, but time also has a direction. Present to past is time backwards and present to future is time forwards.*

T10 Be careful. You are using the word 'direction' in some kind of general sense. When we talk of a vector, we are talking of a direction in space. Time is not associated with any

direction in space, so it is a scalar.

Free and localized vectors

- S11 *That is clear. Let me turn to something else. I see in books that vectors are freely displaced from one point to another (keeping their length and direction constant). Is a vector not supposed to be located at a particular point in space?*
- T11 Ordinarily, in vector algebra, vectors are defined by their magnitude and direction only, and have no association with any particular location in space. Two vectors are equal if both have the same magnitude and direction, regardless of their location. Therefore, for vector operations (addition, subtraction, etc.), there is no error in displacing a vector from one point to another provided you do not change its length or direction.
- S12 *I am still not clear. A force, for example, is a vector quantity. But obviously the effect of a force depends not only on its magnitude and direction but also at what point it is applied.*
- T12 You are right. For instance, the rotational effect of a force on a rigid body depends on

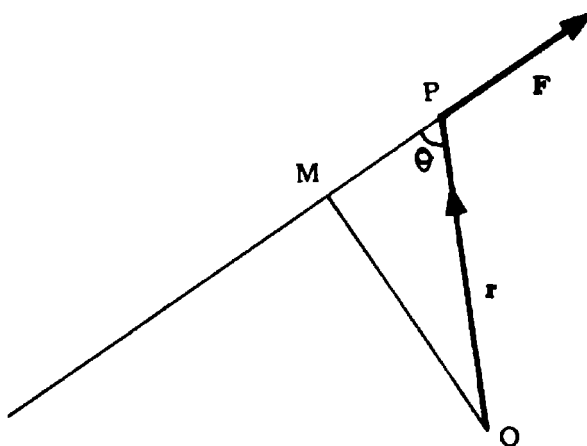


Fig.2.3 The lever arm of the force F acting at P is $OM = r \sin \theta$.

its line of application, because magnitude of torque = magnitude of force $\times$ lever arm, and the latter depends on the line of application of the force (See Fig.2.3).

- S13 *Why do you use the words 'line of application' ? Why don't you merely refer to the point where the force is applied.*
- T13 Because the particular point on the line where the force may be applied is immaterial. The lever arm is simply the distance of the line from the point O around which torque is to be calculated.
- S14 *Is it generally true in mechanics that the effect of a given force depends on its line of application and not on the particular point where it is applied ?*
- T14 No. That is true if you are considering rigid bodies. For a body that is elastic or deformable, the effect of a force also depends on the point where it is applied. For example, in Fig.2.4, two forces of equal magnitude and opposite direction, acting along the same line of application, are exerted on an elastic rod. If the effect of a force

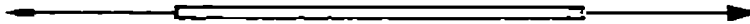


Fig.2.4 Two equal and opposite forces applied at the two ends of an elastic rod do not produce a null effect.

depended only on its line of application, the resultant would be a null force with no effect. But that, clearly, is not true; the length of the rod is greater when the forces act. Thus, in this case, the effect of a force is determined not solely by its line of application, but by its point of application as well.

- S15 *Then you agree with what I said in S12. It means I cannot displace vectors like the way it is shown in books.*
- T15 Vectors which have no association with any particular location in space are called 'free vectors'. Free vectors can always be displaced.

- S16 *Oh, that is begging my question. It just means force, for example, is not a free vector and, therefore, vectors associated with forces cannot be displaced.*
- T16 You are right. Some people like to call a vector which is bound to a particular location in space as 'localized' or 'bound' vector. Force is an example of a localized vector.
- S17 *Still, it is very customary in books to show forces on a body displaced so that they all appear together at one point, and then use the parallelogram law of addition. But you just agreed that force is a localized vector and cannot be displaced.*
- T17 You should always understand that certain things which may not be true in general could be correct in limited contexts. Consider as you said, a body under the action of a number of external forces applied at different points. Now if we are interested only in the translational motion of the body as a whole (i.e. motion of its centre of mass), then it is perfectly correct to displace the forces such that they are all at a point (centre of mass) and add them up by the usual laws of vector addition. In many elementary problems in mechanics, it is the translational motion or translational equilibrium that we investigate. In that context, it is quite correct to treat forces as free vectors. Of course, the moment you are interested in say rotational motion or rotational equilibrium, you cannot displace forces; and in that context force is not a free vector.
- S18 *Can you give me some other example of a localized vector ?*
- T18 There are many examples. Suppose we are given a charge distribution in space. With each point in space, you can associate a vector, namely the force a unit positive charge would experience if placed at that point. This collection of vectors is called electric field. Likewise one can also define magnetic field. Since vectors here are associated with points in space, the vectors representing the electric or magnetic fields are clearly localized vectors.
- S19 *Is there a similar notion for scalars also ? Can scalars also be localized ? Are there scalar fields ?*
- T19 Yes, of course. Inherent in the idea of a field is association of something with each

point in space. If that something is a scalar quantity, then we have a scalar field. If you associate a vector with each point, then we have a vector field. Electric and magnetic fields are instances of vector fields. In an environment where temperature may be varying from point to point, I can talk of a temperature field, which is a scalar field. Likewise we talk of potential (gravitational or electric) as a scalar field.

Multiplication of two vectors

- S20 *I now understand the notion of localized vectors. Let me ask something else. For scalars, we have only one type of multiplication. How is it that we have two types of multiplication for vectors ?*
- T20 Look at it this way. 'Addition' or 'Multiplication' are merely names given to binary operations on a set of objects. (Binary operation means an operation involving two elements of the set.) For scalars, i.e. ordinary numbers, the meanings of these operations are familiar to us. Now vectors are a different set of objects from scalars. For this set, the meanings of 'addition' and 'multiplication' have to be defined afresh. These definitions do not follow from the corresponding definitions for scalars. You already know that the 'addition' of vectors is defined by the parallelogram law. In the same way we need to define the meaning of multiplication of two vectors.
- S21 *O.K. but why two different types of multiplication ?*
- T21 It so happens that besides 'addition' it is possible to have two very useful binary operations on vectors. Do not worry that we use the word 'multiplication' for these operations. What we call these operations is not important. The significant thing is that two useful binary operations can be defined.
- S22 *I do not see why you use words like 'useful definition'. A definition is a definition. It is either correct or incorrect. Either vectors have two types of multiplication, or they do not.*
- T22 There is an important issue here and I believe you need a different orientation in your thinking. Otherwise, at many points in mathematics and physics, you will ask similar

questions and find yourself puzzled. I repeat, the so-called scalar multiplication and vector multiplication are two different and useful definitions of binary operation on vectors (besides addition).

S23 *I still do not get you.*

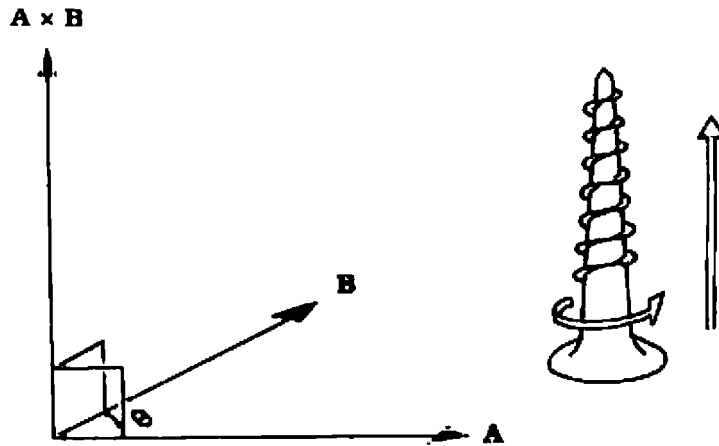


Fig.2.5 Definition of vector product of two vectors.

T23 You must understand that no definition is right or wrong, because a definition is our construct, our choice. But, of course, a definition may be very significant and useful, or it may be trite and useless. The following two binary operations have been found to be useful. From two vectors **A** and **B**, I can construct a scalar $AB \cos\theta$ where A , B are magnitudes of **A** and **B**, and θ is the angle between **A** and **B**. Likewise I can construct a vector whose magnitude is $|AB \sin\theta|$ and whose direction is normal to the plane of **A** and **B**. (The sense of the normal is determined by the motion of the right-handed screw rotated from **A** to **B** through the smaller angle. See Fig.2.5)

(Note, each of these objects involves product of magnitudes of **A** and **B** besides other things. So I call $AB \cos\theta$ as scalar product of **A** and **B** and the vector defined above as vector product of **A** and **B**.)

S24 *What do you mean by the smaller angle between A and B in your definition of vector product?*

T24 The angle between any two directions in space can be given two values : 0 and $2\pi - \theta$. Of these, one will be smaller than (or equal to) π , and the other will be greater than (or equal to) π . In the scalar $A B \cos\theta$ it is immaterial which value I take because $\cos(2\pi - \theta) = \cos\theta$. But in specifying the direction of the vector product of **A** and **B** I must choose one of these angles, because the directions of advance of a right-handed screw rotated through the two angles are opposite. We choose the smaller angle.

S25 *Well then, where do we stand now ?*

T25 We started out trying to define 'multiplication' of two vectors. We have seen that from two vectors we can construct both a scalar and a vector. Next, for convenience, I give a notation for these operations. I denote the object named scalar product by the symbol $\mathbf{A} \cdot \mathbf{B}$. Similarly I denote the object named vector product by the symbol $\mathbf{A} \times \mathbf{B}$. Therefore

$$\mathbf{A} \cdot \mathbf{B} = A B \cos \theta ,$$

$$\mathbf{A} \times \mathbf{B} = A B \sin \theta \hat{n} ,$$

where $\hat{n}$ is a unit vector in the direction of $\mathbf{A} \times \mathbf{B}$ as specified in T23. Sometimes the notations themselves generate new names. Thus the scalar product $\mathbf{A} \cdot \mathbf{B}$ is also called dot product of **A** and **B**, and the vector product $\mathbf{A} \times \mathbf{B}$ is also called cross product of **A** and **B**.

S26 *I am clear about the two definitions of multiplication of vectors, but are these definitions of any use ?*

T26 This is a very legitimate question. That is precisely the orientation I was looking for in your thinking. I hope you have now got the point that the issue in physics often is not the correctness of a definition but its usefulness. Coming back to your question, scalar and vector products are indeed very useful. They appear so frequently in physics that one can hardly live without them.

S27 *Can you give some examples of their usefulness ?*

T27 Consider a force **F** applied over a displacement **d** of a particle. The scalar built out of **F** and **d**, i.e. $\mathbf{F} \cdot \mathbf{d}$ has important physical significance. It denotes the work done

by $\mathbf{F}$ resulting in a change in kinetic energy of the particle. In the same way, the vector built from the position vector $\mathbf{r}$ and momentum $\mathbf{p}$ of a particle, namely the cross product $\mathbf{r} \times \mathbf{p}$, is an important quantity, the angular momentum $\mathbf{L}$. The vector product $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ denotes the torque produced by the force $\mathbf{F}$ around the origin. Both $\mathbf{L}$ and $\boldsymbol{\tau}$ have great significance in physics, especially in problems of rotational motion. In electromagnetism, the vector $\mathbf{E} \times \mathbf{B}$ is related to the radiative energy flow. The volume of a parallelepiped of sides $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ is given by $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. The examples abound.

Tensors

- S28 *I grant you that these definitions of multiplication of vectors are useful. But since, as you said, definitions are our constructs, can't we have some more definitions of multiplication? From 'multiplication' of two vectors $\mathbf{A}$ and $\mathbf{B}$, why do I necessarily have to get a scalar or a vector? Can't I get something else?*
- T28 It is for you to try out and see if you can arrive at any additional useful definitions. But I know at least one more way of 'multiplying' two vectors. From a vector $\mathbf{A}$ with components A_i , $i = x, y, z$ and vector $\mathbf{B}$ with components B_j , $j = x, y, z$ I can construct 9 product quantities $A_i B_j$. These nine quantities together form an object which is neither a scalar nor a vector.
- S29 *What then is this object?*
- T29 This 9-component object is called a tensor (of rank 2). We can denote it by T_{ij} (where $i = x, y, z$; $j = x, y, z$) and identify its components as $T_{xx} = A_x B_x$, $T_{xy} = A_x B_y$, etc.
- S30 *But A_i , B_j are components of vectors $\mathbf{A}$, $\mathbf{B}$. T_{ij} are components of what?*
- T30 One usually works with tensor components only. But if you insist, I can introduce an object $\vec{\vec{T}}$, whose components are T_{ij} . Such an object is called a dyadic. A dyadic is a sort of 'double vector'. To get its components, simply take appropriate scalar product from either side. Thus

$$T_{xy} = \hat{i} \cdot \vec{T} \cdot \hat{j} , \quad T_{yz} = \hat{j} \cdot \vec{T} \cdot \hat{k} , \quad \text{etc.}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along x , y and z axes respectively. The dyadic constructed out of vectors $\mathbf{A}$ and $\mathbf{B}$ may be denoted by $\mathbf{AB}$. Its components are

$$A_x B_x = \hat{i} \cdot \mathbf{A} \mathbf{B} \cdot \hat{i} , \quad A_y B_z = \hat{j} \cdot \mathbf{A} \mathbf{B} \cdot \hat{k}$$

and so on.

S31 *Another question. We have 'addition', 'subtraction' and 'multiplication' of two vectors. Is there any meaning to the division of $\mathbf{A}$ by $\mathbf{B}$?*

T31 If $\mathbf{A}$ and $\mathbf{B}$ are in the same (or opposite) direction, i.e. $\mathbf{A}$ is a scalar multiple of $\mathbf{B}$ ($\mathbf{A} = \lambda \mathbf{B}$), then it is natural to define division of $\mathbf{A}$ by $\mathbf{B}$ as

$$\frac{\mathbf{A}}{\mathbf{B}} = \lambda$$

But for any two arbitrary vectors $\mathbf{A}$ and $\mathbf{B}$ it does not appear possible to give a straightforward definition of division. However, if one means by division of $\mathbf{A}$ by $\mathbf{B}$ as 'something' which 'multiplied' by $\mathbf{B}$ gives you $\mathbf{A}$, then some progress is possible. If $\mathbf{A}$ and $\mathbf{B}$ are related by an expression of the type

$$\mathbf{A} = \vec{T} \cdot \mathbf{B} ,$$

where $\vec{T}$ is a 'double vector', a dyadic, then, in a sense, $\vec{T}$ is what you get when $\mathbf{A}$ is 'divided' by $\mathbf{B}$. In component form, this equation looks like

$$A_i = \sum_j T_{ij} B_j , \quad i = x, y, z$$

which shows that our quest for trying to define $\mathbf{A}$ divided by $\mathbf{B}$ does not quite give us any simple answer: rather it ends up in the notion of tensors.

S32 *We seem to be ending up in these objects called tensors, both while trying to generate new types of multiplication of two vectors and also in our attempt to define division of two vectors. But in the same spirit that you emphasized in T26, is the tensor T_{ij} , the way you have defined, a useful construct at all?*

T32 It is indeed so. Tensor analysis is one of the most useful branches of mathematics

which now underlie basic physical theories. It is especially important when in a theory issues such as, how quantities behave under such and such transformations, which quantities are invariant, whether the given physical laws are the same for different frames of reference, etc. are in focus. But since these matters are somewhat more advanced, I will not dwell on them here.

S33 *Still you can give me some feel about these objects. Are there physical quantities which are tensors ?*

T33 Yes, there are numerous examples of tensors. You will realize from the preceding discussion that whenever in a physical situation a vector quantity **A** is related to another vector quantity **B** that is not necessarily in the same direction as **A**, the 'coefficients' connecting **A** and **B** are going to be tensors.

S34 *Are there situations of that kind ?*

T34 Yes. Consider a dielectric placed in an electrostatic field **E**. The field will produce a polarization **P** in the medium. Now, if the direction of **P** is the same as that of **E**, we may write

$$\mathbf{P} = \chi \mathbf{E} ,$$

where χ is a scalar called the electric susceptibility of the medium. Such a relation is true only for isotropic dielectrics. Some dielectrics (for example, anisotropic crystals), however, exhibit significant direction dependence and in such cases, **P** is not necessarily in the same direction as **E**. For such anisotropic dielectrics, we can write

$$P_i = \sum_j \chi_{ij} E_j \quad i = x, y, z$$

where χ_{ij} is a 9-component object, a tensor of rank 2. It is known as the susceptibility tensor.

Other examples of tensors are the moment of inertia tensor, stress tensor in the theory of elasticity, electromagnetic field strength tensor etc.

Vector nature of area

S35 *Perhaps we have had enough of tensors. Let me ask another little doubt. Is area a scalar or a vector ?*

T35 The magnitude of area is, of course, a scalar quantity. But any element of area is not fully specified by its magnitude alone. We must know how that area element is oriented. For example, a 1 cm^2 area in x-y plane and another 1 cm^2 area in y-z plane

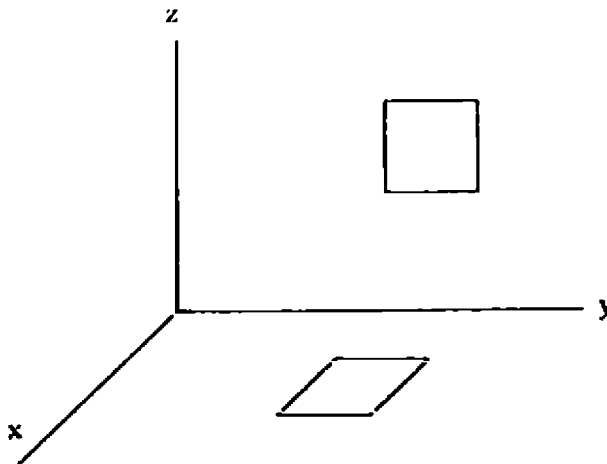


Fig.2.6 Area elements with different orientations.

are not identical things. (See Fig. 2.6) They have the same magnitude, but are oriented differently. So an area element is specified fully only if you give its magnitude as well as its direction; It is therefore a vector quantity.

S36 *But how to specify the 'direction' of an area element ? On the rectangle shown in Fig.2.6. I can draw lines in all kinds of directions. Which of these shows the direction of area ?*

T36 None. The direction of area element is the direction from which you can tell the orientation of the area. Obviously, the orientation is specified if you give the direction of the normal to the area element.

- S37 *I note you use the words 'area element' by which I guess you mean a small, strictly infinitesimal, area. What is the point ?*
- T37 If we are dealing with a finite area in a plane, I can associate with it a definite direction normal to its plane. For a finite non-planar area the direction of normal may differ from point to point. An infinitesimal area, however, can be treated as a planar area and, therefore, has a definite direction for its normal. This is why I have been talking of an area element and not a finite area.[†]
- S38 *But even a planar area does not have a definite direction. The normal can be 'up' or 'down'. Which of the two alternatives does one choose ?*
- T38 You are right about the ambiguity regarding the sense of direction of the vector associated with an area element. There is no intrinsic way to resolve this ambiguity, but it can be handled through some agreed conventions. For an area element of a closed surface, by convention, we take the outward normal as the direction of the vector area element. But there is no 'outward' or 'inward' for an open surface bounded by some curve. In this case I can agree on a convention relating the direction of vector

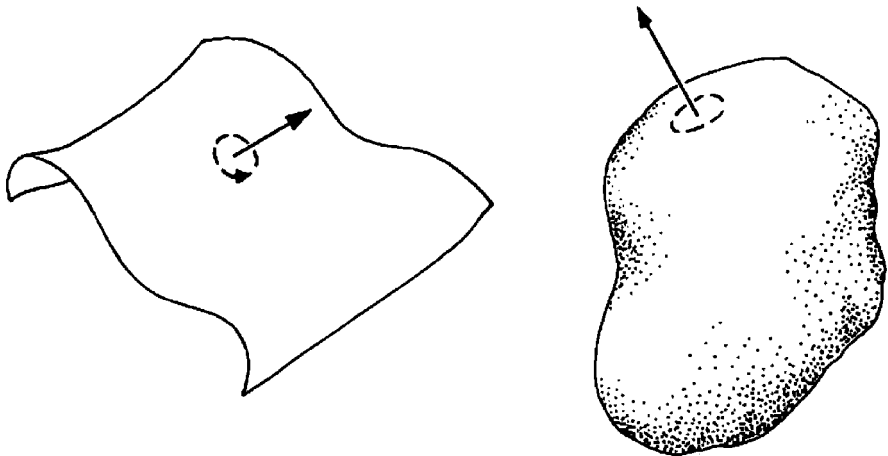


Fig.2.7 Direction of area element of a closed and an open surface.

[†] The vector area of a finite non-planar surface can, however, be obtained by integration over the surface.

area element to the sense of traversing the curve. Thus to decide the direction of an area element we choose a sense of traversing the curve bounding the element. Now if a right-handed screw is rotated in this sense, the advance of the screw gives the direction of the area element. (See Fig.2.7)

S39 *I understand the vector nature of an area element, but it all seems so unnecessary. Why not consider area simply as length times breadth, or something like that? Why must I bother about its direction?*

T39 Depends on what you are doing. If what matters to you is just the magnitude of the area, and not how it is oriented, then your object of interest is a scalar. (The magnitude of a vector is a scalar). Most problems of school mathematics need only the magnitude of the area and that is why you have this lingering impression of area being a scalar. But in many problems, the orientation of the area is equally important. For example, in a running stream the amount of water flowing across an area element obviously depends on the orientation of the area. Similarly, if we want to know the flux of an electric field $\mathbf{E}$ passing through an element of area, both the magnitude and orientation of the area are important. I must then associate a vector $\delta \mathbf{A}$ with that element and write the flux as $\mathbf{E} \cdot \delta \mathbf{A}$.

Pressure and current are scalars

S40 *Let us consider another physical quantity : pressure. Is it a scalar or a vector?*

T40 A scalar obviously.

S41 *It is not so obvious to me. Pressure is supposed to be force divided by area. Now force is a vector, and area in this definition, I suppose, stands for the magnitude of area. A vector divided by a scalar should be a vector.*

T41 Be careful about the precise definition of a quantity, especially when there is some confusion. Consider an area element of magnitude δA . Now this element may be subject to some force. In general, the force may be in any direction. Pressure is defined to be the normal component of the force divided by δA . That is

$$P = \frac{\mathbf{F} \cdot \hat{\mathbf{n}}}{\delta A}.$$

S42 *I see. So I was wrong in defining P as*

$$P = \frac{\mathbf{F}}{\delta A}.$$

Well, the way you have defined, it is clear that pressure is a scalar. But just a little query. If $\mathbf{F}$ is not normal to the element, it will also have a tangential component. What happens to that ?

T42 That component produces another kind of stress, the so-called shearing stress. You see, a body under the action of a general deforming force experiences different kinds of stress, leading to different kinds of strain. The normal component of the force gives rise to 'volume stress' (i.e. pressure) leading to volume change but not any change in shape. The tangential component produces shearing stress leading to shape changes in the body.

S43 *I just got an idea of how pressure might be a scalar. Pressure may be defined by*

$$P = \frac{\mathbf{F}}{\delta \mathbf{A}}.$$

The vector nature cancels in the numerator and denominator and hence P is a scalar. Am I right ?

T43 Completely wrong. Please go over what we discussed in T31 again. Division of $\mathbf{F}$ by $\delta \mathbf{A}$, if you insist on writing the way you have written, makes sense only if $\mathbf{F}$ and $\delta \mathbf{A}$ are in the same direction. If $\mathbf{F}$ and $\delta \mathbf{A}$ have different directions, your definition does not make sense. The words 'vector nature cancels' are particularly objectionable. Vector algebra has its own rules and you should not uncritically take over all the words and operations that you are used to in arithmetic.

S44 *Still, you did try to do something about the division of $\mathbf{A}$ by $\mathbf{B}$ when $\mathbf{A}$ and $\mathbf{B}$ are not in the same direction.*

T44 *I never said there was anything like $\mathbf{A} / \mathbf{B}$ (when $\mathbf{A}$ and $\mathbf{B}$ have different directions)*

but rather used the question to lead you up to tensors.

S45 Here too then $\mathbf{F}$ is a vector and $\delta\mathbf{A}$ is also a vector not necessarily in the same direction. Would relating components of $\mathbf{F}$ to components of $\delta\mathbf{A}$ lead me up to tensors?

T45 Yes, it will. You can guess the equation yourself :

$$F_i = \sum_j P_{ij} \delta A_j, \quad i = x, y, z$$

This, very briefly, is the way one introduces the stress tensors P_{ij} in the theory of elasticity. The details cannot be gone into, but I think you have got the point.

S46 Next, is electric current I a scalar or a vector quantity ?

T46 Let us see the definition of this quantity. First there is a quantity called current density. It is by definition a vector, denoted by $\mathbf{j}$. The direction of $\mathbf{j}$ is the direction of the flow of positive charges. The magnitude of $\mathbf{j}$ at a point is the amount of charge flowing per second through a small area normal to the flow around that point, divided by the magnitude of that small area. It is easy to see that

$$\mathbf{j} = \rho \mathbf{v},$$

where ρ is the charge density at the point, and $\mathbf{v}$ the velocity of the flow.

S47 How is $\mathbf{j}$ related to I ?

T47 The definition of a current always involves a surface, a cross-section of some conductor, etc. Current I across a surface is the total charge flowing across the surface per second. Now, note that at each point on the surface, $\mathbf{j}$ or the direction of flow may not be normal to the given surface. To get the amount of charge flowing across the surface, we must dot $\mathbf{j}$ with the normal $\hat{\mathbf{n}}$ to the surface at the point. Further $\mathbf{j}$ refers to the rate of charge flow per unit area. There is no 'per unit area' in the definition of I . Therefore, we must integrate over the area of the surface, i.e.

$$I = \int \mathbf{j} \cdot \hat{\mathbf{n}} \, da ,$$

which is clearly a scalar.

S48 *I understand this from your definition. But still I can't get rid of the impression that there is a direction to current and so it is a vector.*

T48 To convince you that current is not a vector, let us recall Kirchhoff's first law for steady currents at a junction. Suppose at a junction there are two incoming currents I_1 and I_2 and an outgoing current I_3 . According to this law, $I_3 = I_1 + I_2$. Now this equality is true no matter what the relative directions of the wires carrying currents. This surely cannot be the case if currents were vector quantities. (If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, the equality $C = A + B$ for their magnitudes is not true for arbitrary directions of $\mathbf{A}$ and $\mathbf{B}$; it is true only for the case when $\mathbf{A}$ and $\mathbf{B}$ are in the same direction.)

Rotations

S49 *Let me turn to another thing. If I can associate a magnitude and direction with some quantity, is it necessarily a vector?*

T49 Usually, but not always. For example, rotation is associated with a direction in space (the axis of rotation) and magnitude (the amount of angle of rotation about the axis). Yet rotation is not a vector.

S50 *Why not?*

T50 Because two rotations do not necessarily commute. That is, if R_1 is one rotation and R_2 is another rotation, and if by addition we mean one rotation followed by the other, then you can convince yourself geometrically that in general they do not commute. Symbolically we may express it thus

$$R_1 \oplus R_2 \neq R_2 \oplus R_1 .$$

(If R_1 and R_2 have common axis of rotation then they do commute.)

S51 *So what if they do not commute ?*

T51 Vector addition obeys the parallelogram law and hence is necessarily commutative. Since addition of rotations is not commutative, rotations are not vectors.

S52 *This seems to mean the familiar definition of a vector as given in S1 is not completely adequate. What then is the precise way of defining a vector ?*

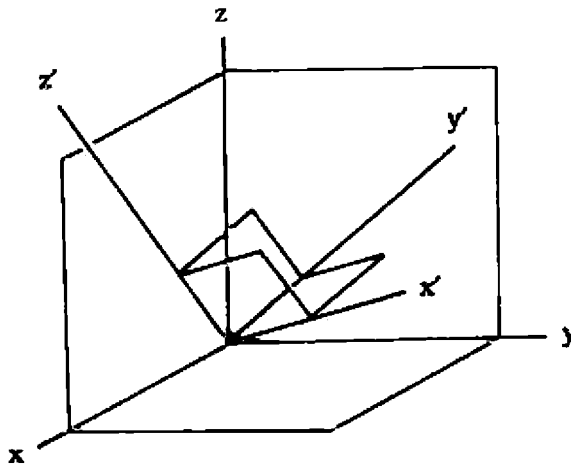


Fig.2.8 Rotation of axes.

T52 To answer you, let me translate the property of having magnitude and direction in space in different terms. Suppose I consider a simple vector, the position vector $\mathbf{r}$ of a point P with reference to some origin O in a co-ordinate system. This vector has components x , y and z along the three axes.[†] Now what do you think will happen if I rotate the axes to some other orientations and call the components of $\mathbf{r}$ along the new axes by x' , y' and z' ? (See Fig. 2.8.)

[†] We use rectangular cartesian frames throughout our discussion.

- S53 *Nothing will happen since you have not rotated the vector r . It will continue to have the same magnitude and direction.*
- T53 That is right. But what about the components of this same vector r along the new axes? How are they related to the old components?
- S54 *I do not know. It will depend on how the axes were rotated: I mean, by what angle and around which direction.*
- T54 That is correct. But let me make life simple for you. Suppose we rotated the original axes around the z -direction by an angle θ as shown in Fig.2.9. What then is the relation between x, y, z and x', y', z' ?

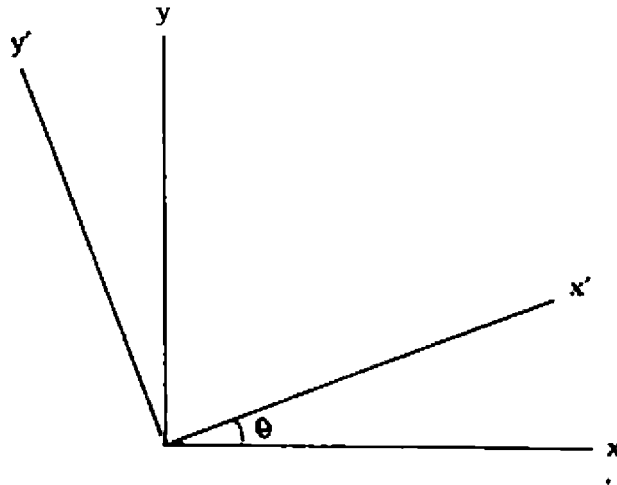


Fig.2.9 Rotation of axes around z -axis.

- S55 *I do not know. All I can easily see is that $z' = z$.*
- T55 Relating x', y' to x, y in this case is not difficult at all. Try it as an exercise in geometry and you will find the following result :

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

S56 *I notice that the old and the new components satisfy the following relation :*

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 .$$

T56 That relation has got to be true, because each side represents square of the length of the vector, and the length of the vector obviously does not change under rotation of axes.

S57 *How about the general case, when even z' is different from z ?*

T57 For rotation in a plane, we just required one angle to specify the rotation. For a general rotation in space, one requires three angles, say α , β and γ . The transformation equations in this case are

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

where the co-efficients R_{ij} are functions of α , β and γ . They are a little complicated so I will not write them down here. But, of course, whatever they may be, the relation given in S56 has got to be true for the same reason as mentioned in T56.

Definition of a vector

S58 *I understand how components of a position vector $\mathbf{r}$ transform as we rotate a frame to a new orientation. But why are you telling me all this ? I asked you a straight question: What is the precise definition of a vector ?*

T58 I am trying to answer just that question. A vector is a three-component object whose components transform under rotations of co-ordinate frames precisely like the components of position vector $\mathbf{r}$.

S59 *Can you repeat that ?*

- T59 **A** is a vector if its components A_x , A_y and A_z in one frame are related to its components A'_x , A'_y and A'_z in a rotated frame in exactly the same manner as in T57, i.e.

$$\begin{pmatrix} A'_x \\ A'_y \\ A'_z \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}.$$

When I say the components of **A** transform in the same manner as the components of **r**, I mean the transformation matrix here is identical to that appearing in T57.

- S60 *That is clear. But what about the commutativity of vector addition? Does it follow from your definition?*
- T60 No. Commutativity follows from the parallelogram law of vector addition, which is a separate requirement. We require that

$$(\mathbf{A} + \mathbf{B})_i = A_i + B_i$$

This ensures that if A_i and B_i transform under rotations of axes as given in T59, $(\mathbf{A} + \mathbf{B})_i$ will also transform similarly.

- S61 *Your technical definitions are clear to me. I am also clear that rotation cannot be associated with a vector because two rotations, in general, do not commute. But I have seen books using vector notation for angular velocity ω . I have also come across the vector relation $\mathbf{v} = \omega \times \mathbf{r}$. How come a rotation (written symbolically as θ) is not a vector but ω is a vector?*
- T61 Look at it this way :

$$\omega = \lim_{\delta t \rightarrow 0} \frac{\delta \theta}{\delta t},$$

i.e. in the definition of ω one does not need a finite rotation θ , but an infinitesimal rotation $\delta \theta$. Now it so happens that two arbitrary infinitesimal rotations $\delta \theta_1$ and $\delta \theta_2$ do commute even though finite rotations do not. Since the definition of ω involves only $\delta \theta$, ω is a valid vector. In other words, we should not think of angular

velocity ω as the time derivative of some vector angle θ , but rather as an infinitesimal rotation vector divided by an infinitesimal time interval.

Conservation is invariance in time

S62 *Let me ask you another thing that sometimes confuses me. Is a scalar quantity always conserved ?*

T62 Be careful again about the words you use in physics. We call a quantity a scalar if it has no direction in space. To put it in the language of transformations that we used for defining a vector, we can say that a quantity is a scalar if it does not change under rotations of coordinate axes. We say a quantity is conserved if it does not change in time, i.e. if its initial value is the same as its later (or earlier) value. The two words mean different things. Sometimes a common word 'invariant' may be used. In that case, you must try to see what it is supposed to mean from the context. A scalar is an 'invariant' under space rotations, a conserved quantity is an 'invariant' in time.

S63 *I get the point. Any examples so that I can absorb it better.*

T63 You can think of so many examples yourself. Here are some. Take kinetic energy of a particle falling under gravity. This is a scalar but not a conserved quantity. Total energy of an isolated system is both a scalar and a conserved quantity. Similarly total charge of an isolated system is both a scalar and a conserved quantity. Thus a scalar may be conserved or may not be conserved. It depends on what the quantity is and what the situation is.

S64 *But is the opposite true ? Is a conserved quantity necessarily a scalar ?*

T64 Not at all. You know angular momentum is a vector. And if there are no external torques, this vector quantity is conserved, i.e. both the magnitude and direction of angular momentum do not change in time in the absence of external torques. So here we have an example of a conserved vector quantity.

Equivalence of rotated frames

- S65 *Let me now ask something else. We are always cautioned that in any equation involving vectors, we should see that the left and right sides of the equation have the same character i.e. if left side is a vector (a scalar), right side should also be a vector (a scalar). Why is this so ?*
- T65 *Is it not obvious ? Suppose I write a stupid thing like a scalar $b = \mathbf{A}$. The left side is just one number, whereas the right side refers to a triplet of numbers. The equality is obviously absurd. In the same way I cannot equate tensor of rank 2 to a vector because the former has 9 components and a vector has only three. And so on.*
- S66 *No, but suppose I write an equation like $b = A_x$ where b is a scalar and A_x is the x -component of a vector. Now there is no problem of the kind you raised. Each side contains just one number. Or I may write, say, $T_{ij} = S_i$. Here T_{ij} is T_j with $j = i$. T_j ($i, j = x, y, z$) being a second rank tensor, and S_i ($i = x, y, z$) is a vector. Here again the number of quantities on either side is the same (equal to 3). Are such equations possible in physics ?*
- T66 *You must realize that it is not merely a question of counting how many quantities there are on each side. The equations you have suggested could appear in some particular frames chosen for a problem. But if you want the equations to be expressed in a form that looks the same in any frame, no matter how it is orientated, then these equations are not appropriate. (Equations having the same form in different frames[†] are sometimes described as 'form invariant' or 'covariant' equations.)*
- S67 *I did not get you. Can you illustrate what you mean by an equation having the same form in different rotated frames ?*
- T67 *Well, an equation like $b = \mathbf{A} \cdot \hat{n}$ (where $\hat{n}$ is determined by some direction in the problem) has the same form in different frames. This is because both sides of the equation have the same character under rotation. (Both are scalars.) The form invariance of the equation is evident also when we write the scalar product in the*

[†] The phrase 'different frames' here, and up to T78, refers to frames which are rotated with respect to each other.

component form

$$b = A_x n_x + A_y n_y + A_z n_z = A'_x n'_x + A'_y n'_y + A'_z n'_z$$

Now if we choose our coordinate axes such that $\hat{n}$ is along, say, x-axis then this equation becomes $b = A_x$. This equation is not wrong; only it is true in a particular frame. It does not exhibit the form invariance that the original equation does.

S68 *Your example of form invariance looks to me rather trivial. In the equation $b = \mathbf{A} \cdot \hat{n}$ each side is a scalar and does not change under rotations anyway. So it is bound to be form invariant.*

T68 You are right. I shall give you a better example. Consider the equations

$$A_x = B_y C_z - B_z C_y,$$

$$A_y = B_z C_x - B_x C_z,$$

$$A_z = B_x C_y - B_y C_x.$$

The quantities occurring here are not scalars; they are components of vectors. The components of these vectors will differ from frame to frame. Yet I claim that these equations are form invariant.

S69 *How do you see that ?*

T69 The way to see that is to make use of the transformation equations given in T59 for any general rotation. For simplicity you may restrict to rotations about z-axis as in T55. You should be able to see that the equations in T68, when expressed in terms of primed quantities using the transformation equations given in T55, lead to

$$A'_x = B'_y C'_z - B'_z C'_y,$$

$$A'_y = B'_z C'_x - B'_x C'_z,$$

$$A'_z = B'_x C'_y - B'_y C'_x.$$

Evidently, the form of these equations is identical to that of the equations in T68. I hope the meaning of 'form invariance' is now clear to you. To repeat, it is not that the quantities remain the same in the two frames (though that happens if the quantities are scalars); it is the form of the relations between the quantities which remain unchanged.

S70 It occurs to me that I need not go to the explicit transformations in T55 to prove the relations in T69. I notice that $(B_y C_z - B_z C_y)$ is the x component of $\mathbf{B} \times \mathbf{C}$. i.e. the equations in T68 could be rewritten as

$$A_x = (\mathbf{B} \times \mathbf{C})_x,$$

$$A_y = (\mathbf{B} \times \mathbf{C})_y,$$

$$A_z = (\mathbf{B} \times \mathbf{C})_z.$$

In other words these equations are nothing but $\mathbf{A} = \mathbf{B} \times \mathbf{C}$. I can now take the components of this vector equation in S' frame and get

$$A'_x = (\mathbf{B} \times \mathbf{C})'_x,$$

$$A'_y = (\mathbf{B} \times \mathbf{C})'_y,$$

$$A'_z = (\mathbf{B} \times \mathbf{C})'_z.$$

which lead to the relations in T69.

T70 Excellent. That is precisely the great advantage of writing equations in vector form. Vector equations are frame-independent. And so if you take their components in different frames, the resulting equations are bound to have the same form, i.e. equations written in vector form automatically guarantee form invariance.

S71 Can you summarize this discussion on form invariance of equations ?

T71 In a form invariant equation, transformation property (under rotation) of the two sides must match. Thus an equation like $C_i = A_i + B_i$ is form invariant because each side transforms similarly. Likewise an equation like $T_{ij} = S_{ij} + A_i B_j$ is form invariant because each side is a tensor of rank two. But an equation like $b = A_x$ or $T_{iy} = S_i$ do not show form invariance. These equations could arise from form invariant equations in some particular frames.

Equations written in vector notation are independent of any frame and guarantee form invariance.

S72 Can you give some examples to consolidate the point ?

T72 Remember the formula for volume of a parallelepiped $V = \mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ in T27. Each side

is a scalar. The formula makes no reference to any frame. If you write the formula in component form in two arbitrary frames S and S' it will obviously have the same form:

$$V = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A'_x & A'_y & A'_z \\ B'_x & B'_y & B'_z \\ C'_x & C'_y & C'_z \end{vmatrix}. \quad (1)$$

However, if one goes to a particular frame where **A** is along the x-axis, one gets

$$V = A_x (B_y C_z - B_z C_y). \quad (2)$$

Eq.(2) is not wrong but is true only in a particular choice of frames. In an arbitrary frame V is not given by Eq.(2). Thus Eq.(2) does not have the form invariance that the original formula shows.

As another example consider the magnetic field of a point dipole. The magnetic field **B** at **r** due to a point dipole of dipole moment **m** located at **r'** is given by

$$\mathbf{B}(\mathbf{r}) = \frac{3\hat{n}(\hat{n} \cdot \mathbf{m}) - \mathbf{m}}{|\mathbf{r} - \mathbf{r}'|^3},$$

where $\hat{n}$ is a unit vector in the direction pointing toward **r** from **r'**. If, however, we choose our coordinate frame such that its origin is at the dipole and the z-axis points in the direction of **m** then the field components are given by

$$B_x(\mathbf{r}) = 3m \frac{xz}{r^5},$$

$$B_y(\mathbf{r}) = 3m \frac{yz}{r^5},$$

$$B_z(\mathbf{r}) = 3m \frac{z^2}{r^5} - \frac{m}{r^3}.$$

Again, this formula, although not wrong, does not have the form invariance that the formula in vector notation guarantees.

S73 *I now understand the notion of form invariance of equations in different frames. But perhaps I should have asked you this in the beginning; why do you demand this form invariance at all?*

T73 *Because I believe that frames S and S' having different orientations of axes must be*

completely equivalent as far as laws of physics or geometry are concerned. There is no preferred orientation of axes in space. So I should always be able to express laws in a form which looks the same in different frames.

S74 *Is that just a matter of faith ?*

T74 In a sense, yes. It is equivalent to saying that space is isotropic i.e. there is no intrinsically preferred direction in space. I suppose experience bears this out. Things do not seem to behave differently in two houses just because they are oriented differently.[†]

S75 *This business of equivalence of rotated frames confuses me a little. Quantities do change from one frame to another. Yet you insist the frames are equivalent.*

T75 Of course, quantities may take different values in different frames. We have seen that in T59. Equivalence of frames means laws of physics make no distinction between frames of different orientation i.e. the relations connecting various quantities do not change their form from one frame to another. In other words physical equivalence of different frames $\Leftrightarrow$ form invariance of mathematical equations which express the laws of physics.

S76 *In practice one does seem to have preferred frames. For example, if I am considering the motion of a charged particle in a constant magnetic field $\mathbf{B}$, it makes good sense to align one of the axes of my frame along $\mathbf{B}$. An arbitrary orientation of axes will be very inconvenient to work with.*

T76 Be careful. Do not equate 'a convenient frame' to 'an intrinsically preferred frame'. For a given problem a frame may be convenient for handling your equations. That does not make this frame intrinsically preferred over others from the point of view of basic laws of physics, and the basic characteristics and correlations of phenomena that these laws describe.

[†] This observation, strictly, implies translational and rotational invariance of the laws of physics.

- S77 *But while working in a convenient frame S, I may not see its equivalence with some other frame S'. In fact I may not be interested in any other frame at all.*
- T77 That may well be so. That is precisely what we saw in T72. When you are working out things in a particular frame, making convenient choices of this and that (without losing generality), plugging in numbers for quantities etc. this equivalence will not stand out. But that does not mean your laws do not respect the equivalence of different frames. It is just that it is not manifest in your procedure. If somebody insists, you must be able to work your steps backwards and show that your basic equations are expressible in a form that shows this equivalence manifestly.

Reflection of axes

- S78 *I have heard about pseudoscalars and pseudovectors. Can you explain what these terms mean ?*
- T78 This has to do with how quantities behave under reflection of axes, i.e. under the transformation

$$x' = -x .$$

$$y' = -y .$$

$$z' = -z .$$

You can easily see that under these reflections, a right-handed system is turned into a left-handed system and vice versa. (See Fig.2.10.)

- S79 *What about these transformations ?*
- T79 You see, under reflection, components of position vector $\mathbf{r}$ transform as given in T78. A vector $\mathbf{A}$ whose components transform similarly under reflection (i.e. $A_i' = -A_i$, $i = x, y, z$.) is called a polar vector. But sometimes one may have a vector $\mathbf{B}$ whose components transform exactly like those of $\mathbf{r}$ under rotations but behave differently under reflection, i.e. $B_i' = +B_i$, $i = x, y, z$. Such a vector is called an axial vector or a pseudovector.

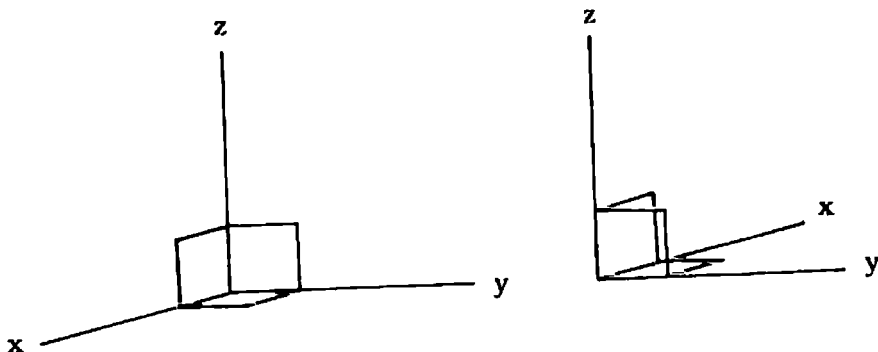


Fig.2.10 (a) A right-handed coordinate system. (b) A left-handed coordinate system.

S80 Do we come across such vectors in practice ?

T80 Yes. First take the linear momentum vector

$$\mathbf{p} = m \frac{d\mathbf{r}}{dt}.$$

$\mathbf{p}$ is clearly a polar vector. Now consider the familiar angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. Under reflection components of both $\mathbf{r}$ and $\mathbf{p}$ reverse their signs. Clearly the components of $\mathbf{L}$ will not change under reflection. So $\mathbf{L}$ is an axial vector. A cross product of two polar vectors is an axial vector. A cross product of an axial and a polar vector is a polar vector. And so on.

S81 And what is a pseudoscalar ?

T81 Consider the dot product of a polar vector $\mathbf{P}$ with an axial vector $\mathbf{Q}$. Clearly $\mathbf{P} \cdot \mathbf{Q}$ is invariant under rotations like any other scalar. Under reflection, however, components of $\mathbf{P}$ change sign but those of $\mathbf{Q}$ do not. Consequently $\mathbf{P} \cdot \mathbf{Q}$ changes sign under reflection, unlike an ordinary scalar. Such a quantity is called a pseudoscalar.

You must have realized how easy it is to decide if something is a scalar or a pseudoscalar; or if it is a vector or a pseudovector. The transformation property under reflection is so simple.

S82 *Yes. But what is the big point ? Why make so much fuss about how quantities behave under reflection of axes ?*

T82 The answer is simple. We want the laws of physics to have the same form as we go from right-handed to left-handed system. To ensure this, we must know how quantities transform under reflection.

S83 *Why do you make that requirement ?*

T83 The answer is similar to what I gave in T73.

Physical equivalence of rotated frames implies that the laws of physics be form invariant under rotations, i.e. the left and the right sides of the equations have the same transformation character under rotations.

Likewise, physical equivalence of reflected frames implies that the left and the right sides of the equations (laws) of physics have the same transformation character under reflection of axes.

S84 *But why do you believe in the physical equivalence of reflected frames ?*

T84 Because we believe that nature has no preferred handedness, i.e. physical laws should not distinguish between left-handed and right-handed frames. (For an exception, see T85.)

S85 *Any examples ?*

T85 The motion of a charged particle in an electromagnetic field described by $\mathbf{E}$ and $\mathbf{B}$ is given by combining Newton's law with Lorentz force equation,

$$\frac{d\mathbf{p}}{dt} = q \mathbf{E} + q \mathbf{v} \times \mathbf{B}.$$

Now the left side is a polar vector. Also we know that $\mathbf{E}$ is a polar vector and $\mathbf{B}$ is an axial vector. (See Box 2.2.) The cross product of the polar vector $\mathbf{v}$ and the axial vector $\mathbf{B}$ is a polar vector. This means both the sides of the equation transform similarly under reflection, which in turn means the equation respects the equivalence of left-handed and right-handed systems.

Box 2.2

Electric field $\mathbf{E}$ at a point $\mathbf{r}$ is determined by the charge distribution described by charge density ρ ,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}') (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'.$$

In this expression, ρ and $|\mathbf{r} - \mathbf{r}'|$ are scalars and $(\mathbf{r} - \mathbf{r}')$ is a polar vector. Clearly, therefore, $\mathbf{E}$ also is a polar vector.

The magnetic field $\mathbf{B}$ at $\mathbf{r}$ is determined by the current distribution, described by current density $\mathbf{j}$, as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r'.$$

The current density is related to charge density ρ and velocity of the flow of charges $\mathbf{v}$ by the relation $\mathbf{j} = \rho \mathbf{v}$. Since ρ is a scalar and $\mathbf{v}$ is a polar vector, $\mathbf{j}$ is a polar vector. It is now clear that the expression for $\mathbf{B}$ involves cross product of two polar vectors, namely $\mathbf{j}$ and $\mathbf{r} - \mathbf{r}'$, and hence $\mathbf{B}$ itself must be an axial vector.

This is true of all equations in electromagnetism. In fact this equivalence is respected by all forces of nature but for an important exception. The exception is the weak interactions of elementary particles which, for example, are responsible for nuclear beta decay.

S86 *One last question for curiosity. I have heard of something called four-vectors. What are these 'animals'?*

T86 Before I tell you about them I want you to note the word transformation that has come up throughout this chapter. Transformation generally means change from the original to something new. In the sense we have been using this word here it means change of axes, or better change of frame of reference, and the resulting change in the values of physical quantities. So far in this chapter we considered two types of transformations, rotations of axes and reflection of axes. But there are other interesting transformations that one can consider. For example, one can consider transformation of quantities between two frames in relative motion. We defined scalars, pseudoscalars, vectors, pseudovectors etc. in terms of their behaviour under rotational and reflection transformations. Analogously we can define 'scalars', 'vectors' etc. in terms of the behaviour under these other transformations.

In special relativity the three space coordinates x, y, z and the time coordinate t of an event transform to new coordinates when we go from one inertial frame S to another inertial frame S' in uniform relative motion. So, in place of the three-component position vector (x, y, z) , we now have a four-component object $(t; x, y, z)$ that transforms to $(t'; x', y', z')$ when we go from S to S' . These transformations are known as Lorentz transformations. A set of four quantities $(A_0; A_1, A_2, A_3)$ which transforms like $(t; x, y, z)$ is a four-vector under Lorentz transformations. You will learn more about them later.

S87 *Can you give me a gist of what you just said ?*

T87 The notion of scalars, vectors and tensors is always with reference to a given type of transformation. Strictly, a statement that something is a scalar or a vector is incomplete. The complete statement is that a given quantity is a scalar or a vector under such and such transformation. In elementary physics one rarely talks of any transformations other than rotations or reflections. Hence this very important point is often missed.

S88 *I will remember that. Shall we come down to something simpler ?*

3

Kinematics

I. One-Dimensional Motion

Instantaneous velocity

S1 *We talk of instantaneous velocity of a particle. I am not sure if I understand clearly what that means.*

T1 *In the course of motion the position of a particle changes. For one-dimensional motion the position of the particle relative to some point (origin) O can be described by a variable x which is a function of time, i.e. $x = x(t)$. The definitions of velocity and acceleration are based on this function.*

S2 *I know them. Velocity is simply displacement $x_2 - x_1$ divided by the time interval $t_2 - t_1$. Acceleration is the change in velocity $v_2 - v_1$ divided by time interval $t_2 - t_1$.*

$$v = \frac{x_2 - x_1}{t_2 - t_1} ; \quad a = \frac{v_2 - v_1}{t_2 - t_1}$$

T2 Can you improve on your answer ?

S3 *I do not understand your question. These are the elementary definitions of velocity and acceleration that are learnt in school. For example, if a train covers a distance of 200 km in 4 hours along a straight track, we say its velocity is 50 km/h in the direction of motion. Or, if a car moving in east to west direction picks up speed from 0 to 10 ms^{-1} in 5 s, its acceleration is 2 ms^{-2} in east to west direction. What is wrong with these definitions?*

T3 Well, tell me, can velocity and acceleration in your definitions depend on time?

S4 *Bodies need not have uniform velocity or acceleration. These could, of course, change with time.*

T4 Then to which time does v or a as you have defined in S2 refer to: t_1 or t_2 or some other time?

S5 *Well, let me see. It could be t_1 . It may well be t_2 , or $(t_1+t_2)/2$. Frankly, I am confused.*

T5 The confusion arises because you are loosely using the words 'velocity' and 'acceleration'. In your examples, strictly speaking, you should use the words 'average velocity or average acceleration over the given interval of time'. In other words, the definitions you have given in S2 are correct definitions of average velocity and average acceleration over an interval t_1 to t_2 :

$$v_{\text{ave}} = \frac{x_2 - x_1}{t_2 - t_1}; \quad a_{\text{ave}} = \frac{v_2 - v_1}{t_2 - t_1}.$$

Of course, if v is constant over the interval t_1 to t_2 then v_{ave} is obviously the same as v . Similarly, if a is constant over the time interval then a_{ave} is the same as a .

S6 *Then what are the defining relations for v and a if they are varying in time?*

T6 In that case you will have to define velocity at an instant, the so-called instantaneous velocity $v(t)$. Similarly you will need to define instantaneous acceleration $a(t)$.

S7 *This instantaneous thing confuses me. To talk of velocity we need to talk of some displacement and the time interval required for the displacement. When we talk of velocity at an instant, we are not referring to any time interval or displacement. What is the meaning of velocity if you have not covered any distance ?*

T7 You must get this absolutely clear. The meaning of instantaneous velocity $v(t)$ can be understood only in a limiting sense, and hence requires calculus. You see, a finite displacement $x_2 - x_1$ covered in a finite time interval $t_2 - t_1$ gives v_{ave} .

$$v_{ave} = \frac{x_2 - x_1}{t_2 - t_1}.$$

Now let $t_2 \rightarrow t_1$ i.e. $t_2 - t_1 \rightarrow 0$. Clearly, x_2 will also tend to x_1 i.e. $x_2 \rightarrow x_1$ or $x_2 - x_1 \rightarrow 0$. But you must have learnt in calculus that the limit of a ratio that has both the numerator and the denominator going to zero can be a well-defined number. This number denotes the velocity at time t_1 (which is the same as the velocity at t_2 since $t_2 - t_1 \rightarrow 0$).

$$v(t_1) = \lim_{t_2 \rightarrow t_1} \frac{x_2 - x_1}{t_2 - t_1}.$$

The right side of this equation is the derivative of $x(t)$ with respect to t at $t = t_1$, i.e.

$$v(t)|_{t=t_1} = \left. \frac{dx}{dt} \right|_{t=t_1}.$$

Now, this is true whatever t_1 we may take. So we can just as well write

$$v(t) = \frac{dx}{dt}.$$

By analogous reasoning, the instantaneous acceleration $a(t)$ is defined as

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}.$$

S8 *Sometimes we talk of a particle instantaneously or momentarily at rest during its motion. How can a particle ever be at rest if it is undergoing motion ?*

Box 3.1

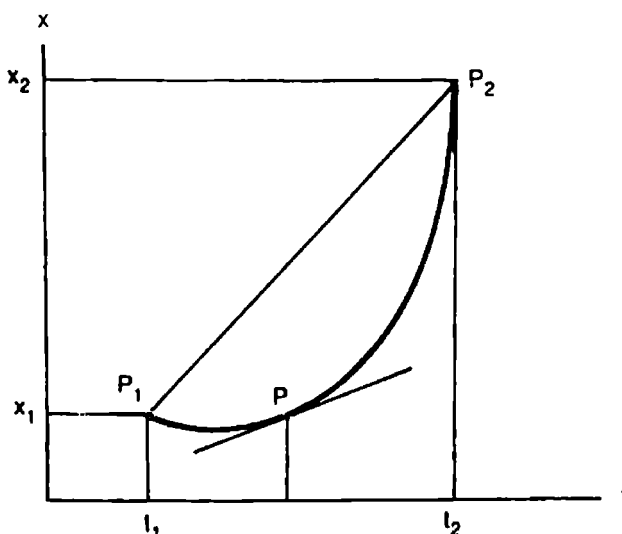


Fig.3.1 An x - t graph of motion of a particle.

The difference between average velocity and instantaneous velocity can be clearly understood if we look at the graph of a function $x(t)$.

Let the x - t graph for motion of a certain particle be as shown in Fig.3.1. The average velocity of the particle over the time interval $t_2 - t_1$ is $(x_2 - x_1)/(t_2 - t_1)$. This is just the slope of the chord P_1P_2 .

On the other hand, the instantaneous velocity of the particle at a point, say P , is dx/dt which is the slope of the tangent to the x - t graph at that point. This, clearly, is different from the slope of the chord in general.

T8 It simply means that a particle, during the course of its motion, has zero instantaneous velocity at some particular instant(s). This is quite possible, because velocity of a particle in motion is, in general, a function of time, and may be zero at some instant(s). Put another way, the graph of function $x(t)$ may be such that at some points the tangent may have zero slope. At such points the instantaneous velocity of the particle will be zero. (See Fig.3.2)

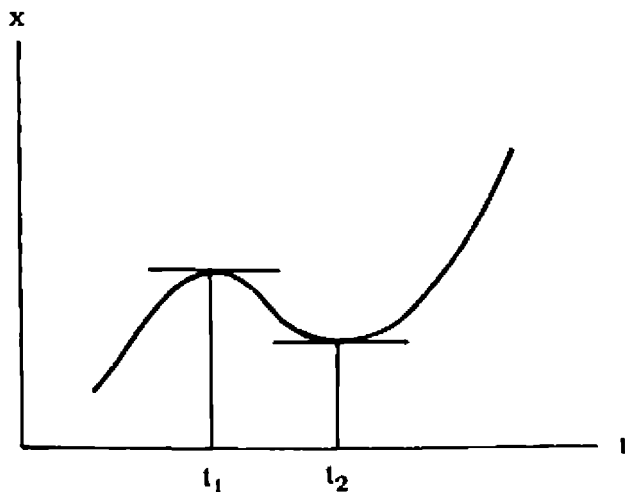


Fig.3.2 Another x - t graph for motion of a particle. The instantaneous velocity is zero at the instants t_1 and t_2 .

- S9 Any examples where a particle is at rest during the course of its motion at some instant(s) of time ?
- T9 There are so many examples that are familiar to you. A ball thrown up is momentarily at rest when it is at the topmost point. An oscillating pendulum is at rest at those instants of time when it is at its either extreme. A point on the rim of a wheel that rolls perfectly without slipping is at rest at the instant it is in contact with the ground.
- S10 When a particle is instantaneously at rest, i.e. its velocity is zero at some instant, is its acceleration also zero at that instant ?
- T10 Not necessarily. It is a serious error if you think so. Just go over the examples in T9. At the instant the ball is at the topmost point its velocity is zero but its acceleration continues to be g , the acceleration due to gravity. For an oscillating pendulum the velocity is zero at either end but its acceleration there is not zero; in fact the magnitude of acceleration is maximum at the end points. In the same manner the acceleration of a point on the rim of a rolling wheel, when the point is in contact with ground, is nonzero. (The last example is not of one-dimensional motion.)

S11 But how can that be ? I mean if $v = 0$, isn't $a = dv/dt$ also zero ?

T11 If v is zero, or has any constant value, at all times over an interval then certainly a is zero over the interval. But if $v = 0$ only at a particular instant, a may not be zero. If a function $v(t)$ is zero at $t = t_0$, it by no means implies that its derivative is zero at

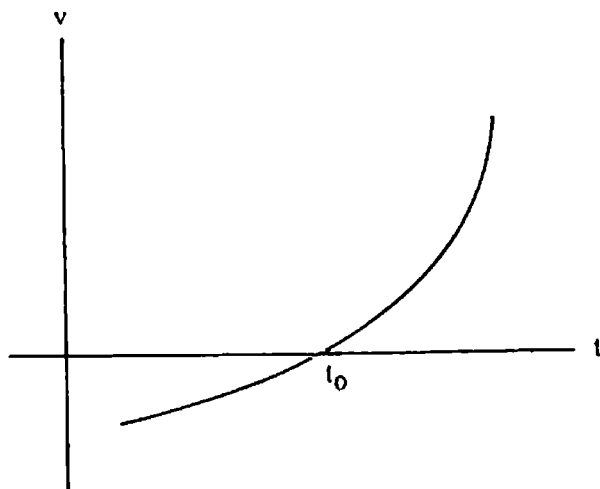


Fig.3.3 A v - t graph for motion of a particle.

that point. This can be clearly seen from a v - t graph as shown in Fig.3.3. The graph intersects the t -axis at $t = t_0$ so that $v(t_0) = 0$. The slope of the graph, $dv/dt = a$, at that point is, however, not zero.

Signs of kinematic quantities

S12 Let us turn to the standard relations of kinematics in one-dimension.

$$v = u + at ; \quad s = ut + \frac{1}{2}at^2 ; \quad v^2 - u^2 = 2as .$$

I want to be clear about the signs of the quantities here. Are the quantities u and v supposed to be positive ?

T12 You must realize that all the quantities occurring in these equations are algebraic i.e. they may be positive or negative (or zero).

S13 *Then how to decide on the signs of these quantities in a given problem ?*

T13 For a given problem choose any one direction to be the positive direction and stick to this choice throughout. Velocities and accelerations along the chosen direction are positive and those in the opposite direction are negative. In the above relations s stands for the displacement from the initial to the final position. It is positive if s is along the chosen direction and negative otherwise.

Sometimes s is confused with the magnitude of displacement. To avoid this confusion, let us write x in place of s .

S14 *But what about the signs of the unknown quantities ?*

T14 You have to do nothing about them to begin with. Simply start with the kinematic relations in S12 and put in the known quantities with appropriate signs. The solutions will give you the signs of unknown quantities. Interpret them by the same sign convention. Thus, if the solution gives the final velocity to be positive it means it is along the chosen direction. If it is negative, it is opposite.

S15 *What if a problem is not of acceleration but of retardation ? Surely, then, I must change a to $-a$ in the kinematic relations.*

T15 This is not necessary at all. The situation of uniform retardation is equally well included in the kinematic relations in S12. You do not need to change these expressions. All you must do is to put the numerical value of a with proper sign.

S16 *You mean I should take the value of a to be positive for speeding up and negative for slowing down. Right ?*

T16 Be careful. Speeding up means acceleration is in the same direction as the instantaneous velocity. Slowing down means acceleration is in the opposite direction

to the velocity.[†]

Now, often in simple problems where motion is unidirectional, one takes the positive direction to be the direction of velocity and then what you say is correct. But when in a problem the direction of motion changes, you have to be careful.

For example, consider a ball thrown up which returns after some time. During the upward journey the ball slows down; its acceleration is in the opposite direction to that of velocity. During the downward journey it speeds up; its acceleration is in the same direction as that of its velocity.

Now if I choose the vertically upward direction to be the positive direction throughout the motion, the value of acceleration is negative throughout, though during downward journey the speed of the ball is increasing.

S17 *That is clear enough. Yet when it comes to solving problems, usually I take the relations*

$$v = u + at ; \quad x = ut + \frac{1}{2}at^2 ; \quad v^2 - u^2 = 2ax .$$

for acceleration, and

$$v = u - at ; \quad x = ut - \frac{1}{2}at^2 ; \quad v^2 - u^2 = -2ax$$

for retardation. That is the way it is given in some books too. Is there anything wrong?

T17 Nothing wrong really. You are in fact employing the same kinematic relations in either case. All you have done is to choose the direction of motion to be the positive direction and used the symbol a for the magnitude of a .

I repeat : the symbol a in S12 stands for acceleration which may have any sign; the symbol a in your relations (S17) stands for magnitude of acceleration which cannot be negative. To avoid confusion you should use the symbol $|a|$ in your relations instead of a .

[†] This is true in one dimension. In three dimensions speeding up means acceleration vector has a positive component along velocity vector.

S18 All right. But for the actual solving of problems does it matter whether I use relations in S12 or those in S17 ?

T18 If the problem involves unidirectional motion then there is little difference. but if the motion changes direction. It is better to use relations in S12 because, then we do not have to worry about splitting a problem into one of acceleration and one of retardation.

S19 Can you give an example ?

T19 Consider a stone thrown upward with an initial speed of 50 m s^{-1} from a cliff 375 m high from the ground. After what time will it hit the ground ? (Take acceleration due to gravity to be 10 m s^{-2} .)

S20 It is easy. I shall first calculate the time the stone takes to go up. also calculate the height it reaches using the relations for retardation. Then I shall calculate the time it takes to come down to the ground using the relations for acceleration. I will add up the two times to get the answer :

Upward motion

$$u = 50 \text{ ms}^{-1},$$

$$\therefore t = 5 \text{ s.}$$

$$v = 0 \text{ ms}^{-1},$$

$$\text{Now use } v^2 - u^2 = -2ax,$$

$$\text{Use } v = u - at,$$

$$\text{i.e. } 0 - 2500 = -20x,$$

$$\text{i.e. } 0 = 50 - 10t,$$

$$\therefore x = 125 \text{ m}$$

Downward motion

$$u = 0 \text{ ms}^{-1},$$

$$\text{Use } x = ut + \frac{1}{2}at^2,$$

$$x = 125 + 375 = 500 \text{ m.}$$

$$\text{i.e. } 500 = 5t^2,$$

$$\therefore t = 10 \text{ s.}$$

So the stone will hit the ground after $5 + 10 = 15 \text{ s}$.

T20 Your answer is correct. But I can solve the problem more efficiently using the relations in S12. I must choose some direction (up or down) to be the positive direction and stick to it throughout the problem. Let us choose upward direction to be positive. Then the given quantities are

$$u = +50 \text{ m s}^{-1}, \quad x = -375 \text{ m}, \quad a = -10 \text{ m s}^{-2}.$$

$$\text{Use the relation } x = ut + \frac{1}{2}at^2,$$

so that $-375 = 50 t - 5 t^2$.

This gives $t = + 15 \text{ s}$ or $- 5 \text{ s}$.

Since we are interested in finding when the stone hits the ground after it is thrown, we should take the positive solution. It agrees with your answer. I consider this method better because I did not have to worry about the upward and the downward parts of the motion separately. I got the answer from just one equation.

- S21 *I agree with you, but I was a little surprised when you put $x = - 375 \text{ m}$ in the solution above. x represents the height of the cliff. Should I not put it equal to 375 m ?*
- T21 Be careful about the meaning of each quantity in the kinematic relations. Remember x stands for the displacement from initial to final position of the object. This displacement is downward and we have taken the upward direction as the positive direction in the problem. So here x is a negative number.
- S22 *I am now clear about the signs of x , u , v and a but what about the sign of t ? I guess it must always be positive. Isn't that the reason you ignored the solution $t = - 5 \text{ s}$.*
- T22 No. t in kinematic equations can be positive or negative.
- S23 *But how can time be negative ?*
- T23 It is simply a question of which instant of time you choose to call zero. If I take this instant of time when you are reading this line as $t = 0$, all the events before this happened at negative times and all the events after this instant will occur at positive times. In other words, if present is taken to be $t = 0$, all past has $t < 0$ and all future has $t > 0$.
- S24 *I understand that time as such can be assigned negative values. but my question is whether the kinematic relations are true for $t < 0$.*

- T24 The kinematic relations in S12 are true in the entire time domain during which the acceleration a is constant. The sign of t is immaterial.
- S25 *Then why did you ignore the solution $t = -5\text{ s}$?*
- T25 In the kinematic relations of S12 note that at $t = 0$, $x = 0$ and $v = u$ i.e. $x(0) = 0$ and $v(0) = u$. Thus when we used these relations for our problem, the instant of time the stone is thrown up from the cliff corresponds to $t = 0$. The stone will hit the ground after that instant, so the time it hits the ground must be positive.
- S26 *But what is the meaning of the solution $t = -5\text{ s}$? Surely it should have some interpretation.*
- T26 It has no interpretation for the given problem. We do not know what the situation is before the stone is thrown up, i.e. for $t < 0$. If the stone was at rest in the hands of the person who threw it at $t = 0$, we have $x = 0$ for $t < 0$. Thus for $t < 0$ the problem does not correspond to one with uniform acceleration $= -10\text{ m s}^{-2}$. After it hits the ground i.e. for $t > 15\text{ s}$ and $x = -375\text{ m}$, again it is not a problem of uniform acceleration $= -10\text{ m s}^{-2}$. Hence the kinematic relations with $a = -10\text{ m s}^{-2}$ are valid, in the given problem, only for the time interval $0 \leq t \leq 15\text{ s}$. Any solutions outside this interval must therefore be rejected.
- S27 *Then what do we make of the solution $t = -5\text{ s}$?*
- T27 This solution will be perfectly applicable for a problem in which $a = -10\text{ m s}^{-2}$ in the interval that includes $t = -5\text{ s}$ and which has the same conditions at $t = 0$ as in the original problem.
- S28 *I do not understand. Can you clarify ?*
- T28 Suppose I frame a problem like this : a stone thrown up from the ground passes the edge of a cliff 375 m high, with an upward speed of 50 m s^{-1} . Taking the instant it passes the cliff to be $t = 0$, what is the instant it was thrown up and at what instant will it return to the ground ?

S29 *How do you solve this problem ?*

T29 Exactly as in T20. The equation to solve is identical to what we had before. But now both the solutions are interpretable. Both times correspond to $x = -375$ m. $t = -5$ s is the instant the stone was thrown up from the ground and $t = 15$ s is the instant it returns to the ground.

S30 *Can you please consolidate everything that we have discussed so far ?*

T30 i) For problems on one-dimensional motion with uniform acceleration the kinematic relations are :

$$v = u + at ; \quad x = ut + \frac{1}{2}at^2 ; \quad v^2 - u^2 = 2ax .$$

Note, in these equations u corresponds to velocity at $t = 0$ and the displacement x is zero at $t = 0$.

ii) Do not change the algebraic form of these equations from situation to situation.

iii) For a given problem choose any convenient direction to be the positive direction and stick to the choice throughout the solution of the problem.

iv) Having made this choice, substitute the known data for u , v , x , a etc. In so doing do take care whether to put a positive number or a negative number for them. Put a positive value when the given quantity is along the positive direction that you have chosen, a negative value if it is in the opposite direction.

v) Solve for the unknowns. The quantities will come out with appropriate signs. Interpret the signs of the quantities that you obtain as in (iv) i.e. if the quantity turns out to be positive, it is along your chosen positive direction; if it turns out to be negative it is in the opposite direction.

vi) Be reminded that the kinematic relations above are valid only for constant a . If a is given to be a constant only over an interval t_1 to t_2 , then values of t outside this interval are unacceptable.

Origin in space and time is a matter of choice

S31 *I want to know one thing. In a given problem which instant of time should I take to be $t = 0$? And which point should be taken as the origin, i.e. $x = 0$?*

T31 You will no doubt agree that what matters really is the elapsed time and the distance covered in that time. Which particular time you call $t = 0$ and which particular point in space you take as the origin is entirely a matter of choice.

Any instant of time during motion can be taken to be $t = 0$. $t = 0$ does not necessarily correspond to the beginning of the motion. (See T28.) It is customary to call u in the relation $v = u + at$ as 'initial' velocity. The word 'initial' here can be misleading. To avoid possible confusion always think of u above as v at $t = 0$.

For example, in the problem in T28 we took the edge of the cliff as $x = 0$ and the instant the stone passes the edge of the cliff as $t = 0$. But we could just as well take the instant when the stone is thrown from the ground as $t = 0$ and the ground as $x = 0$. Or else we could take the instant the stone is at the topmost point to be $t = 0$ and the topmost point as $x = 0$. Indeed we could take any point on the path as $x = 0$ and the instant the stone passes this point (in a specified direction) to be $t = 0$.

S32 *Let us see if we get the same answer if we make some different choice.*

T32 Suppose I choose the first alternative, i.e. $x = 0$ on the ground and $t = 0$ when the stone is thrown up. Then, when the stone passes the edge of the cliff it has $x = 375$ m and $v = +50$ ms⁻¹. Use of the relation $v^2 - u^2 = 2ax$ gives u , the velocity with which the stone is thrown from the ground, to be 100 ms⁻¹. Let t' be the time the stone passes the edge of the cliff. The relation $v = u + at$ gives t' to be 5 s. Next, let us use $u = 100$ m s⁻¹ and $x = 0$ in the relation $x = ut + \frac{1}{2}at^2$. The resulting quadratic equation, $0 = 100t - 5t^2$, has two solutions : $t = 0$ corresponds to the initial instant of throw and $t = 20$ s to the instant the stone returns to the ground.

S33 *We seem to be getting different answers from what we obtained in T29.*

T33 The absolute values of the times of throw and return to the ground will, of course, not agree with our previous answers because of the shift of zero of time. You must note, however, that the elapsed time between any two events comes out to be the same by either method. For example, the time between the throw and return to the ground is 20 s in either case.

S34 *In all the discussion above we have taken $x = 0$ when $t = 0$. Is that necessary ?*

T34 Yes, if we use the kinematic equations of the form given in T30. This condition, as you can see, is built-in in these relations. Of course, it is possible to modify these equations so that they can be used even when our reference point does not correspond to $t = 0$, $x = 0$.

S35 *How does one modify them ?*

T35 In our kinematic equations t stands for the time elapsed from the instant $t = 0$ when velocity of the object is u . Similarly x represents the displacement from the point $x = 0$. Let us choose $t = t_0$ and $x = x_0$ as our reference, instead of $t = 0$ and $x = 0$; u , now, denotes the velocity at $t = t_0$ rather than at $t = 0$. For notational uniformity let us replace u by v_0 so that velocity at $t = t_0$ is denoted by v_0 .

With the new choice, the elapsed time is $t - t_0$ and the displacement is $x - x_0$. Hence the kinematic equations modify to

$$v = v_0 + a(t - t_0) .$$

$$(x - x_0) = v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2 .$$

$$v^2 - v_0^2 = 2 a(x - x_0) .$$

You can check explicitly here that at $t = t_0$, $x = x_0$ and $v = v_0$ i.e. $x(t_0) = x_0$ and $v(t_0) = v_0$.

S36 *What, then, is the form of the one-dimensional kinematic relations that I should finally remember ?*

- T36 You may remember the relations in T35. It is best to view them as relations connecting values of the two functions $x(t)$ and $v(t)$ at any time t to their values at $t = t_0$. What you must realize is that there is no special status to t_0 in these equations. By that I mean t and t_0 are any two instants of time. To highlight this point let us write t_0 as t_1 and t as t_2 and rewrite the relations above as

$$v(t_2) = v(t_1) + a(t_2 - t_1) .$$

$$x(t_2) - x(t_1) = v(t_1)(t_2 - t_1) + \frac{1}{2} a(t_2 - t_1)^2 .$$

$$v(t_2)^2 - v(t_1)^2 = 2 a (x(t_2) - x(t_1)) .$$

The relations are true for any t_1 and t_2 which belong to the interval of time during which the motion has uniform acceleration a .

Variable acceleration

- S37 You have been emphasizing that the kinematic relations are valid for constant acceleration. What happens in case of time-varying acceleration ? Can I simply put in a time dependence in a , i.e. use $a(t)$ in place of a in the same relations ? I mean, are the following relations O. K. for the case of time-varying acceleration ?

$$\left. \begin{aligned} v(t) &= v_0 + a(t)(t - t_0) . \\ x(t) &= x_0 + v_0(t - t_0) + \frac{1}{2} a(t)(t - t_0)^2 . \\ v(t)^2 &= v_0^2 + 2 a(t)(x(t) - x_0)^2 . \end{aligned} \right\} \quad ?$$

- T37 No. If you know a general result or relation, you can, of course, always go to a special case of the general result. But the converse is not true. You cannot go from a special case to the general case in the adhoc manner that you have adopted.

- S38 Well, then, how do I go about treating the general case ?

- T38 That is an important question. Actually, what you must realize is that kinematic relations such as those in T35 follow from the definitions of velocity and acceleration.

So to derive kinematic relations for the time-varying case simply go back to these definitions :

$$v = \frac{dx}{dt} ; \quad a = \frac{dv}{dt} .$$

The definition of a can be put in an alternative form also,

$$a = \frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx} .$$

S39 *Well, how do you get the kinematic relations from these definitions ?*

T39 Before I answer that, note two things.

i) Definitions of v and a are differential equations: quantities on either side of these equations refer to the same time t .

ii) The kinematic relations (in T35), as already emphasized, relate functions $x(t)$ and $v(t)$ to their values at a particular time $t = t_0$.

Obtaining (ii) from (i) is what one means by integrating differential equations, or solving them with given initial (or boundary) conditions.

S40 *O. K. then. how does one obtain (ii) from (i) ?*

T40 Consider first the simplest situation. A body is at rest at a point x_0 at some instant t_0 , and its acceleration is zero throughout some time interval t_1 to t_2 that includes the time instant t_0 .

S41 *But that is not the situation we are talking about. We are talking about a body moving with varying $v(t)$ and perhaps with varying $a(t)$.*

T41 Remember what I said in T37. A simple situation that is a special case of a general situation is always included in the latter. So you should be able to handle the simple situation with the general method.

S42 *All right. Please proceed.*

T42 Let us see what is given. We know that $a = 0$ throughout the time-interval t_1 to t_2 . Further we have what are called initial conditions, $x(t_0) = x_0$ and $v(t_0) = 0$.

S43 *And what do we have to find out ?*

T43 We must find out x and v for all t in the given interval. This is easily done by integrating the defining relations.

Start with $dv/dt = a$. Now $a = 0$, therefore $dv/dt = 0$ for all t . Clearly, $v(t)$ must be a constant, $v(t) = k$.

It is also given that $v(t_0) = 0$. The constant k must, therefore, be zero. Hence $v(t) = 0$.

Now use the definition of velocity $dx/dt = v$. Since $v(t) = 0$ as already proved, we have $dx/dt = 0$. This gives $x(t) = \text{constant}$. But it is given that $x(t_0) = x_0$. Therefore $x(t) = x_0$.

We have thus solved the problem obtaining $v(t) = 0$ and $x(t) = x_0$.

S44 *I do not see what you have got out after all this. I knew already that the body would be at rest at x_0 all the time.*

T44 You are right. I went through the exercise just to show that solving the differential equations in T38 for this simple case leads to the expected results.

Let us now consider another simple situation. A body is at position x_0 and has velocity v_0 at time t_0 . The acceleration of the body is zero throughout the interval t_1 to t_2 which includes t_0 .

S45 *Well, let me try to imitate what you did in T42 and T43. First we are given that $a = 0$ throughout the time interval. Further, the initial conditions are $x(t_0) = x_0$ and $v(t_0) = v_0$.*

T45 That is correct. What do you do next ?

S46 From $dv/dt = a$ we get $dv/dt = 0$, which gives $v(t) = \text{constant}$. Using the information $v(t_0) = v_0$ we can conclude that this constant must be v_0 . Hence $v(t) = v_0$.

T46 Very good. Go on.

S47 Next I use the relation $dx/dt = v$. With v that we have obtained, this relation becomes $dx/dt = v_0$.

Integration of this equation gives $x(t) = v_0 t + C$, where C is a constant. Now I shall use the other piece of information, namely $x(t_0) = x_0$. Substitution gives $x_0 = v_0 t_0 + C$, which gives $C = x_0 - v_0 t_0$. This, finally, gives me the result $x = v_0 t + x_0 - v_0 t_0$.

T47 Excellent. Let me just rewrite your results : $v(t) = v_0$ and $(x - x_0) = v_0(t - t_0)$. These are the relations that you know anyway for uniform motion. So the method gives expected results in this simple situation also.

Let us, now, consider the motion of a particle with uniform acceleration. In this case $a(t) = a$, where a is a constant. The initial conditions can be taken as $x(t_0) = x_0$ and $v(t_0) = v_0$.

Now $dv/dt = a$. Since a is a constant, integration of this equation gives

$v(t) = at + \alpha$, where α is a constant.

Use of the initial condition $v(t_0) = v_0$ gives α to be $v_0 - at_0$. Therefore,

$v(t) = (v_0 - at_0) + at = v_0 + a(t - t_0)$.

Next, $dx/dt = v(t) = v_0 + a(t - t_0)$.

Integrating both the sides we get

$x(t) = (v_0 - at_0)t + \frac{1}{2}at^2 + \beta$, where β is a constant.

At $t = t_0$ this relation gives

$$x_0 = (v_0 - at_0)t_0 + \frac{1}{2}at_0^2 + \beta,$$

which in turn determines β . When β , thus obtained, is substituted back in the expression for $x(t)$ we obtain

$$x(t) = x_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2.$$

So we have got exactly the kinematic relations for constant acceleration that you are familiar with (See T35).

- S48 *This seems to be a very clear way of deriving kinematic relations. I now appreciate that the kinematic relations are nothing but the solutions of the defining equations for v and a with given initial conditions. But many books give much simpler derivation of these relations. The relation $v(t) = v_0 + a(t - t_0)$ is obtained straight from the definition of acceleration*

$$a = \frac{v - v_0}{t - t_0}.$$

To obtain the relation for displacement one starts with the definition of average velocity

$$v_{ave} = \frac{x - x_0}{t - t_0}.$$

Next we take

$$v_{ave} = \frac{v_0 + v}{2} = v_0 + \frac{1}{2}a(t - t_0),$$

to obtain

$$(x - x_0) = v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$$

Is this derivation correct ?

- T48 This is a perfectly correct procedure provided a is constant.

S49 *Just where has the fact that a is constant gone into in the above derivation ?*

T49 At two places.

First, I have repeatedly stressed that $(v - v_0)/(t - t_0)$ is the definition of average acceleration, not instantaneous acceleration. But, of course, if a is constant it is the same thing as a_{ave} .

Second, you must realize that the formula $v_{\text{ave}} = (v_0 + v)/2$ that you have employed above is true if a is constant, but not for general motion.

S50 *That surprises me. Isn't the average of two numbers always calculated by adding them and dividing the sum by two ?*

T50 Here we are not talking of average of two numbers; we are talking of average of a quantity over an interval. Simple-minded relations like $v_{\text{ave}} = (v_0 + v(t))/2$ and $a_{\text{ave}} = (a(t_0) + a(t))/2$ are not true for arbitrary motion.

S51 *This seems to be an important point. Please elaborate.*

T51 I repeat what I have said. The relations

$$v_{\text{ave}} = \frac{x - x_0}{t - t_0} ; \quad a_{\text{ave}} = \frac{v - v_0}{t - t_0}$$

are always true: in fact they are definitions of v_{ave} and a_{ave} as I have indicated by three lines. But the relations

$$v_{\text{ave}} = \frac{v_0 + v(t)}{2} ; \quad a_{\text{ave}} = \frac{a_0 + a(t)}{2}$$

are not true in general.

S52 *Then how does one calculate these averages in general ?*

T52 For v_{ave} , begin with the definition $dx/dt = v(t)$. Integration of this relation gives

$$x(t) = \int_{t_0}^t v(t') dt' + \text{constant} .$$

Now, at $t = t_0$, $x = x_0$ which gives the constant equal to x_0 .

$$\therefore x(t) - x_0 = \int_{t_0}^t v(t') dt' .$$

This gives

$$v_{ave} = \frac{1}{t - t_0} \int_{t_0}^t v(t') dt' .$$

In the same way

$$a_{ave} = \frac{1}{t - t_0} \int_{t_0}^t a(t') dt' .$$

[In fact average of any quantity $f(x)$ over an interval x_1 to x_2 can be obtained as

$$f_{ave} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} f(x) dx .$$

S53 Does the formula for v_{ave} given in T52 yield $(v_0 + v(t))/2$ for the case of constant a ?

T53 Yes, it does as shown below.

For constant a we have $v(t) = v_0 + a(t - t_0)$. Substituting this in the formula for v_{ave} given in T52, a straightforward integration gives

$$v_{ave} = v_0 + \frac{1}{2} a(t - t_0) = (v_0 + v(t))/2 .$$

S54 That is clear. Let us now return to my original question. What are the kinematic relations in the case of general time-dependent acceleration $a(t)$?

- T54 Here, then, comes the anti-climax. In the general case I cannot go far beyond the defining relations $v = dx/dt$ and $a = dv/dt$ themselves. I can, of course, write from T52

$$x(t) = \int_0^t v(t') dt' + x_0 .$$

$$v(t) = \int_0^t a(t') dt' + v_0 .$$

but these are nothing like the simple algebraic relations that you have in the case of constant acceleration.

You must appreciate though that, for any given $a(t)$, the problem of finding out the kinematic relations has been reduced to doing the integrals appearing above. Thus, although we do not have kinematic relations for arbitrary $a(t)$, we have a systematic procedure to find out these relations for any given $a(t)$. Of course $a(t)$ has to be simple enough so that the integration is possible.

- S55 *There is one thing that I noticed when you were trying to solve the differential equations $dx/dt = v$ and $dv/dt = a$. You always had two initial conditions $x(t_0) = x_0$ and $v(t_0) = v_0$. What is really behind this ?*

- T55 What you must realize is that acceleration alone does not determine x as a function of time completely. That is, one can have different types of motion under the same acceleration.

For example, motion under gravity is characterized by a uniform downward acceleration. But, of course, the initial position and initial velocity of a particle can be anything. I might throw a ball up from the ground or from a cliff. Or, I might just drop a stone from a certain height or shoot it down with some initial velocity. In short, if you simply specify the acceleration, it is not enough to determine what $x(t)$ and $v(t)$ are. One must have two additional pieces of information, the so-called initial conditions, such as the value of $x(t)$ and $v(t)$ at some instant $t = t_0$.

- S56 *I understand that from the examples you gave. But how are you sure that, in general after specifying a , two initial conditions are enough ?*

- T56 That follows from the mathematics of differential equations. The equation $d^2x/dt^2 = a$ is a linear second-order differential equation for $x(t)$. Mathematics tells us that for such an equation two initial conditions, such as $x(t_0)$ and $\left. \frac{dx}{dt} \right|_{t=t_0}$, are necessary and sufficient to determine $x(t)$.

Another way of looking at it is that we have two linear first-order differential equations $dx/dt = v$ and $dv/dt = a$. The second equation needs one initial condition (the value of v at $t = t_0$) to determine $v(t)$ completely. Having obtained $v(t)$, the first equation needs one more initial condition (the value of x at $t = t_0$) to solve for $x(t)$. Either way the answer is the same. We need two initial conditions to get $x(t)$ if a is specified. (Note, there is nothing wrong in specifying x at one instant and v at another instant.)

- S57 *There is another thing I wanted to ask when you were clarifying the difference between the average velocity and instantaneous velocity. Does a body with varying v ever possess velocity equal to v_{ave} during the course of its motion?*

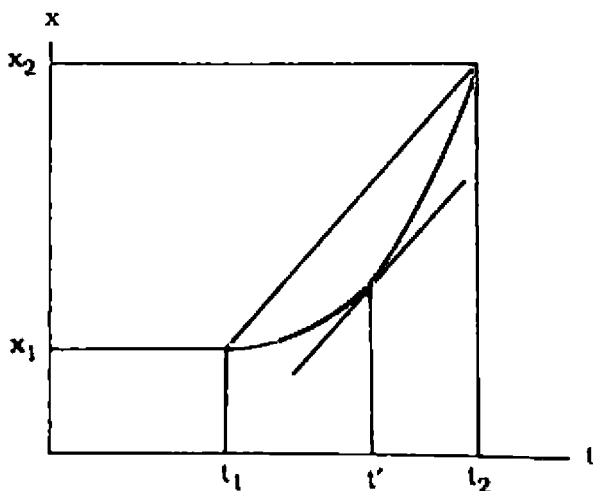


Fig.3.4 An x - t graph for motion of a particle.

- T57 What you are asking is whether v_{ave} can equal $v(t)$ at some instant of time within the interval? The answer to this question is: yes, this is always true. You can understand this if you plot the x - t graph of motion. Suppose the graph for a certain motion is as shown in Fig.3.4. Over the interval t_1 to t_2 , $v_{\text{ave}} = (x_2 - x_1)/(t_2 - t_1)$, which is the slope of the chord connecting the end points of the interval. The instantaneous velocity of the particle at time t is dx/dt , i.e. the slope of the tangent of the x - t graph at that instant. Intuitively it is clear that for any differentiable curve (that is, a curve without breaks and kinks) you will always find a point on the curve where the tangent is parallel to the chord. This is proved rigorously in any calculus text and the result goes by the name of mean value theorem. Hence there will always be some point $t = t'$ within the interval where

$$v(t') = \left. \frac{dx}{dt} \right|_{t=t'} = \frac{x_2 - x_1}{t_2 - t_1} = v_{\text{ave}}.$$

Likewise, provided v - t graph of motion is differentiable, there will always be some point t' within the interval such that

$$a(t') = \left. \frac{dv}{dt} \right|_{t=t'} = \frac{v_2 - v_1}{t_2 - t_1} = a_{\text{ave}}.$$

- S58 Will the point t' be at the middle of the interval t_1 to t_2 necessarily?

- T58 No, not necessarily. It might happen to be so in some cases. For example, consider the case of uniform acceleration. In this case x - t graph is a parabola and the distance $x_2 - x_1$ travelled in time interval $t_2 - t_1$ is given by the relation

$$(x_2 - x_1) = v_1(t_2 - t_1) + \frac{1}{2} a(t_2 - t_1)^2.$$

We can substitute this in the definition of average velocity

$v_{\text{ave}} = (x_2 - x_1)/(t_2 - t_1)$ to obtain

$$v_{\text{ave}} = v_1 + \frac{1}{2} a(t_2 - t_1).$$

Now using the relation

$$v_2 = v_1 + a(t_2 - t_1)$$

we get $v_{\text{ave}} = (v_1 + v_2)/2$.

On the other hand,

$$v(t) \Big|_{t=\frac{t_1+t_2}{2}} = v_1 + a \left(\frac{t_1+t_2}{2} - t_1 \right) \\ = \frac{v_1 + v_2}{2} .$$

Thus, in the case of uniform acceleration, it is indeed true that v_{ave} over any interval t_1 to t_2 is equal to the instantaneous velocity at $t = (t_1 + t_2)/2$. But, of course, this will not be true in general.

x - t graphs

S59 *By the way, we are talking of linear motion and yet you are considering all kinds of curves for the x-t graphs. How ?*

T59 Never confuse the shape of the x-t graph with the actual path of a particle in space. This is such an obvious point that it hardly needs any explanation. Yet so many times, unawares, students take x-t graphs for the shape of the path in space. For one-dimensional uniformly accelerated motion the x-t graph is parabolic. That by no means implies that the particle is moving along a parabolic curve; the motion is one-dimensional i.e. along a straight line.

Since you have made this kind of error, let me ask you a question. Is there anything wrong with the graph in Fig.3.5?

S60 *Well, nothing seems to be wrong. The particle is moving in a circle in an anticlockwise sense.*

T60 So you have instantly forgotten what I just said.

S61 *Oh, it is an x-t graph and not the path of a particle in space. Well, let me see. The graph is wrong because t is shown to be negative.*

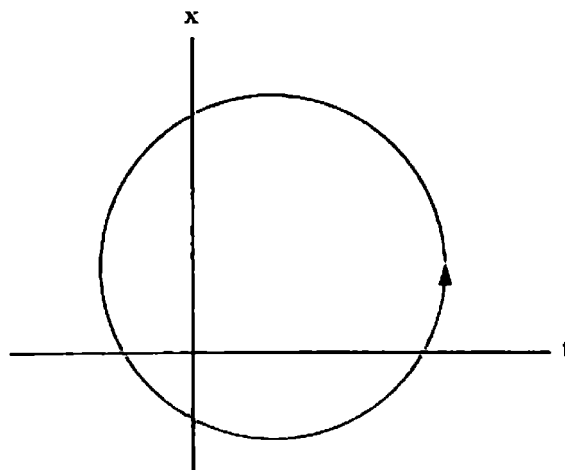


Fig.3.5 An alleged x - t graph.

- T61 There is absolutely nothing wrong with that. I have explained earlier that t can be positive, negative or zero.
- S62 *I now see what is wrong with this graph. For a given t , $x(t)$ is not unique. A particle cannot possibly be at two different locations at the same time. Hence the given graph is absurd.*
- T62 Excellent. But what is that arrow doing on the graph ?
- S63 *I do not know.*
- T63 When you show the trajectory of a particle in space, an arrow does make sense. It shows the direction of traversal of path with increasing time. On an x - t graph it obviously makes no sense. So never put an arrow on an x - t graph.

Displacement and path length

S64 *Let us turn to something else. We use words like 'distance', 'path length', 'displacement'. Is path length or distance travelled the same as the magnitude of displacement ?*

T64 *No, not always. But let us not confuse things by using too many words. Let us agree to use only the term 'path length' and say that by 'distance' we always mean path length. Some people might differ in the use of these terms and might mean by 'distance' the magnitude of displacement. This is the reason I should like to stick to only one term : path length.*

S65 *Yes, but isn't path length the same thing as magnitude of displacement ? For example, I go from a point x_1 to point x_2 as in Fig.3.6. Obviously displacement = $x_2 - x_1$ and*

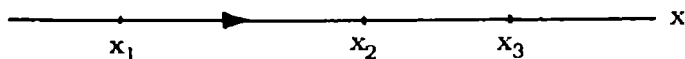


Fig.3.6 A one-dimensional path of a particle.

therefore magnitude of displacement = $|x_2 - x_1|$. Also, the path length = $|x_2 - x_1|$. The two are clearly equal.

T65 *If during the motion you have not changed direction, then what you say is true. But what if I do not go from x_1 to x_2 directly; instead I go from x_1 to x_3 and then turn back to come to x_2 ? In this case the displacement from x_1 to x_2 is still $(x_2 - x_1)$ and the magnitude of displacement is still $|x_2 - x_1|$. But the path length, i.e. the length of the actual path traversed is clearly $|x_3 - x_1| + |x_3 - x_2|$. Obviously this is greater than the magnitude of the displacement.*

S66 *From the way you have explained, it seems that displacement has to do only with the positions of the particle at the beginning and at the end of the time interval of interest.*

It has nothing to do with the actual path taken by the particle in order to complete the journey. Is that right ?

T66 Perfectly. The path length, of course, depends on the actual path traversed by the particle. It is equal to the magnitude of the displacement if you have followed a straight path without any change in direction, not otherwise. As an extreme example I may start from x_1 and return to the same point following any kind of path I choose. The magnitude of displacement, then, is zero, but of course the path length is not zero. (I can experience that last statement in the aching of my legs!)

Average speed and average velocity

S67 *What can we say about average speed and magnitude of average velocity ?*

T67 It is really the same point.

$$\text{Average velocity} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\begin{aligned} \therefore |\text{Average velocity}| &= \frac{|x_2 - x_1|}{t_2 - t_1} \\ &= \frac{\text{magnitude of displacement}}{t_2 - t_1} \end{aligned}$$

On the other hand

$$\text{Average speed} = \frac{\text{path length}}{t_2 - t_1}.$$

Now, path length $\geq$ magnitude of displacement.

$\therefore$ average speed $\geq$ magnitude of average velocity.

The equal sign holds only when, during motion on a straight line, you have not changed direction.

S68 *What about instantaneous speed and magnitude of instantaneous velocity ?*

T68 *They are always equal.*

S69 *Why ?*

T69 *Well, instantaneous quantities are defined in terms of vanishingly small intervals of time and vanishingly small displacements. During such small intervals, the length of the path covered is the same as the magnitude of the displacement. (The length of a vanishingly small arc is the same as that of the corresponding chord.) Consequently, the instantaneous speed of a particle is the same as the magnitude of the instantaneous velocity.*

S70 *Let me put the result in T67 symbolically. If I denote $|\delta x|$ by δs , then $v = dx/dt$ and $|v| = ds/dt$. The displacement in an interval t_1 to t_2 can be obtained as*

$$\int_{t_1}^{t_2} dx = \int_{t_1}^{t_2} v dt .$$

whereas the path length covered in the same interval is given by

$$\int_{t_1}^{t_2} ds = \int_{t_1}^{t_2} |v| dt .$$

Since path length $\geq$ magnitude of displacement,

$$\int_{t_1}^{t_2} |v| dt \geq \left| \int_{t_1}^{t_2} v dt \right| .$$

Dividing by $t_2 - t_1$, I obtain

$$|v|_{ave} \geq |v_{ave}| .$$

Isn't that strange ?

- T70 Well, it just means that the order of the two operations involved, the operation of taking average over an interval and the operation of taking magnitude of a quantity, cannot be reversed. When this happens one says that the two operations do not commute.
- S71 *We have been talking of one-dimensional motion. But the general motion of a body in physical space must be three-dimensional. What, then, is the relevance of studying motion in one dimension ?*
- T71 You are right. Bodies do move in three-dimensional space. But in some situations motion along one direction may be important or relevant, motion along other two directions being uninteresting or unimportant. For example, motion of a train along a straight track is strictly a three-dimensional motion. But one may not be interested in how the train wobbles slightly along the breadth of the track or how it oscillates up and down on its suspension. We may simply be interested in its motion along the straight track for which we can employ the relations for one-dimensional motion.

II. Two and Three-Dimensional Motion.

Vector kinematic relations

- S72 *Let us turn to kinematics in two and three dimensions. In what important ways is it different from one-dimensional kinematics.*
- T72 The key additional complication is due to the directionality of quantities like displacement, velocity, acceleration etc.

S73 *But these quantities have direction in one-dimensional case too.*

T73 True, but in one dimension there are only two directions possible. So we can take care of the direction of a quantity by the use of positive and negative signs. This will not work for two or three dimensions because here, obviously, the number of possible directions is infinite.

S74 *All right. What are the kinematic relations for constant acceleration in three-dimensional case ?*

T74 The generalization is straightforward. The one-dimensional relations that you are familiar with are now true for each cartesian component of position, velocity and acceleration.

$$v_x(t) = v_{0x} + a_x(t - t_0) ,$$

$$x(t) = x_0 + v_{0x}(t - t_0) + \frac{1}{2}a_x(t - t_0)^2 ,$$

$$v_x(t)^2 = v_{0x}^2 + 2a_x(x(t) - x_0) ;$$

Similar relations hold for the y and the z components.

S75 *I suppose I can write these relations more compactly in vector notation as*

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}(t - t_0) ,$$

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{2}\mathbf{a}(t - t_0)^2 ,$$

$$v(t)^2 = v_0^2 + 2\mathbf{a} \cdot (\mathbf{r}(t) - \mathbf{r}_0) .$$

T75 That is right. That in fact is the best way of expressing the kinematic relations in two or three dimensions. This is so since these relations relate position, velocity and acceleration vectors per se without any reference to a coordinate system.

S76 *But how do you arrive at these vector kinematic relations ?*

T76 That is no mystery. Remember what I said for one-dimensional case. All kinematic relations follow from the definitions of velocity, acceleration etc. So you must be clear about the definitions of velocity and acceleration vectors in terms of the position vector.

S77 *How do you define them ?*

T77 Velocity vector is simply the rate of change of position vector. If $\mathbf{r}(t)$ is the position vector at time t and $\mathbf{r}(t+\delta t)$ is the position vector at $t+\delta t$, then the instantaneous velocity $\mathbf{v}(t)$ is given by

$$\mathbf{v}(t) = \lim_{\delta t \rightarrow 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t} = \frac{d\mathbf{r}}{dt}.$$

Likewise, the instantaneous acceleration vector is defined as the rate of change of velocity vector

$$\mathbf{a}(t) = \lim_{\delta t \rightarrow 0} \frac{\mathbf{v}(t + \delta t) - \mathbf{v}(t)}{\delta t} = \frac{d\mathbf{v}}{dt}.$$

S78 *I am confused about the meaning of velocity vector at a certain time t . After all to define velocity there must be some displacement in space over a certain time interval. Similarly to define acceleration I must see how velocity changes over some time interval and divide the change by the time interval.*

T78 You are back to the same confusion that you showed earlier. Please confront this barrier regarding instantaneous velocity and acceleration and try to overcome it.

Whatever you said above refers to average velocity and average acceleration over a time interval.

$$\mathbf{v}_{ave} = \frac{\mathbf{r}(t_2) - \mathbf{r}(t_1)}{t_2 - t_1}; \quad \mathbf{a}_{ave} = \frac{\mathbf{v}(t_2) - \mathbf{v}(t_1)}{t_2 - t_1}.$$

Instantaneous velocity and instantaneous acceleration at some time $t = t_1$ are the limits of these expressions as the interval $t_2 - t_1$ tends to zero. That leads us to the definitions given in T77. In short, if you are clear about the notions of limits and derivatives in calculus then $\mathbf{r}(t)$, $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are simply different functions of time related by equations in T77.

S79 *How does one get kinematic relations from these definitions ?*

T79 As for the one-dimensional case, we can integrate these equations for constant acceleration with suitable initial conditions.

Start with the equation

$$\frac{dv}{dt} = \mathbf{a} .$$

For constant $\mathbf{a}$ the solution of this equation is

$$\mathbf{v}(t) = \mathbf{a}t + \boldsymbol{\eta}$$

Using the initial condition $\mathbf{v}(t_0) = \mathbf{v}_0$ the constant of integration $\boldsymbol{\eta}$ can be evaluated. This gives

$$\mathbf{v}(t) = \mathbf{v}_0 + \mathbf{a}(t - t_0) .$$

Next, integrating this equation and using the initial condition $\mathbf{r}(t_0) = \mathbf{r}_0$ we get

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0(t - t_0) + \frac{1}{2}\mathbf{a}(t - t_0)^2 .$$

Eliminating t between the two equations we can obtain a third relation

$$v(t)^2 = v_0^2 + 2\mathbf{a} \cdot (\mathbf{r}(t) - \mathbf{r}_0) .$$

S80 *For one-dimensional case we needed two numbers to specify the initial conditions. How many do we require here ?*

T80 Although $\mathbf{r}_0$ and $\mathbf{v}_0$ appear to be two quantities, remember that they are vectors. They in fact stand for six numbers. That is as it should be because the two vector equations in T77, in fact, are six first-order differential equations. We know a first-order equation requires for its solution the specification of one initial condition. So we need six initial conditions in all.

S81 *You said the kinematic relations are true for constant acceleration. In uniform circular motion we have constant acceleration equal to v^2/r . Does that mean I can apply relations in S75 to this case ?*

- T81 Be careful. The kinematic relations are true for constant acceleration vector $\mathbf{a}$. This means both the magnitude and the direction of $\mathbf{a}$ should be constant. In uniform circular motion the magnitude of $\mathbf{a}$ is constant but not its direction. The direction of $\mathbf{a}$ is along the radius towards the centre and it is continuously changing during motion. The kinematic relations in S75 are, therefore, not applicable to uniform circular motion.
- S82 *Does the direction of the acceleration vector in uniform circular motion depend on the sense of rotation namely clockwise or anticlockwise ?*
- T82 Obviously not. What appears clockwise to you will appear anticlockwise to somebody facing you. The inward direction of acceleration vector will be common to both.
- S83 *Is the acceleration vector radially inward even when motion along a circle is not uniform ?*
- T83 No. The acceleration now has two components, one along the tangential direction and the other along the radially inward direction. The tangential component describes how the angular velocity of the object changes. The radially inward (centripetal) component of acceleration is, however, given by the same formula v^2/r where v is the instantaneous speed.
- S84 *Are the vector kinematic relations in S75 true for uniform acceleration as well as uniform retardation ? Or must I replace $\mathbf{a}$ by $-\mathbf{a}$ when motion is retarded ?*
- T84 You must realize that $\mathbf{a} = d^2\mathbf{r}/dt^2$ is the definition of acceleration. It is true no matter whether the components of $\mathbf{a}$ along some axes are positive or negative. Replacing $\mathbf{a}$ by $-\mathbf{a}$ in this definition for the case of retarded motion is absurd. In the same way the kinematic relations in S75, which are obtained from this definition, are true as long as all the components a_x , a_y , a_z are constant no matter what their signs. You should not change the algebraic form of these relations but only substitute positive or negative numbers for the components of $\mathbf{a}$ depending on what the situation is.
- S85 *For non-uniform acceleration is it correct to replace constant $\mathbf{a}$ by $\mathbf{a}(t)$ in the kinematic relations ?*

- T85 No. The kinematic relations are obtained by solving the defining relation $\mathbf{a} = d^2\mathbf{r}/dt^2$. Now, this relation is correct for arbitrary $\mathbf{a}$ but it is only when $\mathbf{a}$ is constant that the solution of this equation with initial conditions gives you the kinematic relations in S75. For arbitrary $\mathbf{a}(t)$ integration of the defining relation, if possible at all, will not give the relations you have suggested. Thus, your adhoc procedure is wrong.

Acceleration and trajectory

- S86 *For motion under constant acceleration $\mathbf{a}$, is it correct to say that the trajectory of a particle is always a straight line along the direction of $\mathbf{a}$?*

- T86 Not true. All that you can say is that the component of velocity along the direction of $\mathbf{a}$ changes while the component perpendicular to $\mathbf{a}$ remains constant. If this latter component is zero initially, then it remains zero throughout the motion and there is no motion normal to $\mathbf{a}$, i.e. the particle travels on a straight line (along or opposite to the direction of $\mathbf{a}$). If the component of initial velocity normal to $\mathbf{a}$ is non-zero, the trajectory is a parabola.

For example, a stone thrown without any horizontal velocity moves along a vertical line (along or opposite to the direction of gravity), but a stone thrown at an angle travels along a parabola.

- S87 *Well then, for motion with variable $\mathbf{a}$ is the trajectory of the particle necessarily curved?*

- T87 No. A particle may have varying $\mathbf{a}$ and yet can travel along a line in the same direction. For instance, a vehicle moving along a straight highway may accelerate at one time and decelerate at another time, which means its acceleration is not uniform even though the motion is along a straight line.

Another familiar example is a linear simple harmonic motion in which acceleration varies in magnitude as well as direction.

- S88 What can you say about acceleration vector if motion of a particle is given to be along a straight line?
- T88 All that you can conclude is that the components of acceleration transverse to the straight line are zero. (If they were not zero the particle would acquire transverse velocity components.) Nothing can be concluded about the acceleration component along the line. This component may or may not vary in magnitude and sign.

Displacement and path length

- S89 For motion in space, is the magnitude of displacement vector equal to the path length?

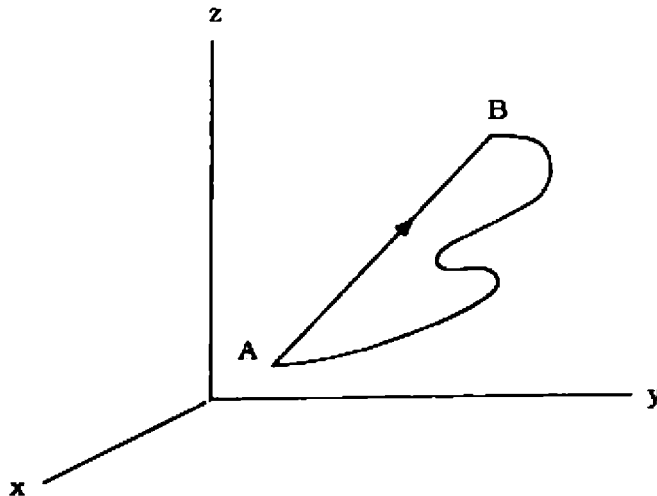


Fig.3.7 A trajectory of a particle in space.

- T89 No, not in general. As shown in Fig.3.7, a particle may travel from a point A to a point B along any curved path. The displacement vector from A to B is along the straight line joining the two points; the magnitude of displacement is just the length of the line AB. The path length is the length of the actual curve followed by the particle. Obviously, therefore, the magnitude of the displacement is less than or

equal to the path length. The equality holds only when the particle travels along the straight line AB without any turning back during motion.

S90 *Why did you use the last phrase : without turning back ?*

T90 The particle may travel along a line but if it changes its direction back and forth then the path length is greater than the length of line AB. This is why, even for one-dimensional motion, one must distinguish between path length and magnitude of displacement. Only when motion is along a straight line in a fixed direction is the path length equal to the magnitude of displacement vector.

Average speed and average velocity

S91 *For two or three-dimensional motion, is average speed the same as magnitude of average velocity ?*

T91 No. As we saw this is not even true in one dimension. Average speed is given by the total path length divided by the time required to travel the path. On the other hand, for average velocity we are not concerned with the actual path. It is just the displacement vector from the initial to the final position that is needed.

Now, the magnitude of displacement vector is either less than or equal to the total path length as already explained. Hence average speed is not the same thing as magnitude of average velocity. Only when motion is along a straight line in a fixed direction are the two quantities equal. In all other cases average speed is greater than the magnitude of average velocity.

S92 *What is the relation between instantaneous speed and magnitude of instantaneous velocity ?*

T92 They are equal to each other.

- S93 *How is it that instantaneous speed = magnitude of instantaneous velocity, but average speed $\geq$ magnitude of average velocity ?*
- T93 For the same reason that I explained for one-dimensional case. Definitions of instantaneous quantities need infinitesimal path or displacement. Now, the length of a finite portion of a curve is greater than the length of the chord (See Fig 3.7) but on an infinitesimal portion the difference between the two is vanishingly small. Hence the result.

Relative velocity

- S94 *Another question. Often some problems in mechanics involve determination of relative velocity which amounts to subtracting two velocity vectors. Can you explain the rationale behind it ?*
- T94 Let us think in terms of concrete problems.

Suppose it is raining on a windy day. The downward velocity of raindrops is 50 ms^{-1} and the wind velocity is 10 ms^{-1} in east to west direction. What is the direction in which a boy standing on ground should hold his umbrella ?

Now suppose the wind stops blowing and the boy rides on a bicycle with the same speed i.e. 10 ms^{-1} and in the same direction i.e. east to west. What is the direction in which he should now hold his umbrella ?

- S95 *I suppose the answers are the same. Because the velocity of the wind in the first problem and that of the boy in the second problem are identical.*
- T95 No. This is something you should be very clear about. In the first problem two velocities are given in a frame and we need to find the resultant velocity in the same frame. As seen from Fig.3.8 (a), the boy should hold his umbrella at an angle $\tan^{-1}(1/5)$ with the vertical towards east. This problem is not a problem of relative motion.

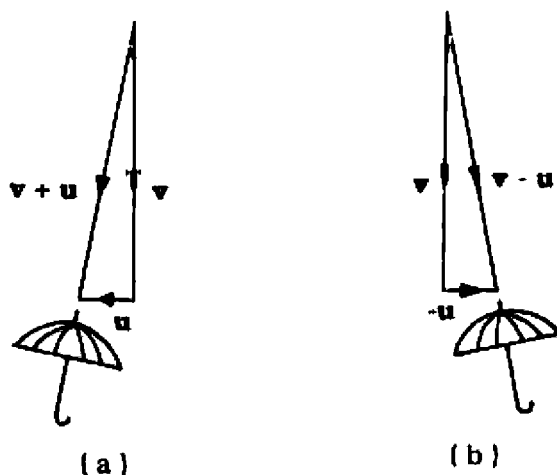


Fig.3.8 (a) Resultant velocity of raindrops in the ground frame. (b) Velocity of raindrops relative to the bicyclist's frame.

In the second situation the velocity of rain in the ground frame is given and what is needed is the velocity of the rain in the frame of reference of the bicyclist. This, therefore, is a problem of relating velocities of the same object as viewed from two different frames in relative motion.

S96 And how does one relate them ?

T96 Suppose an object has velocity $\mathbf{v}$ relative to a frame S and velocity $\mathbf{v}'$ relative to frame S'. S' is moving with a velocity $\mathbf{u}$ with respect to S. From Box 3.2 it is seen that $\mathbf{v}' = \mathbf{v} - \mathbf{u}$.

Now here $\mathbf{v}$ is 50 ms^{-1} vertically downwards and $\mathbf{u}$ is 10 ms^{-1} east to west. Subtracting $\mathbf{u}$ from $\mathbf{v}$ we get $\mathbf{v}'$ as shown in Fig.3.8 (b). In other words the boy should hold the umbrella at an angle $\tan^{-1}(1/5)$ with the vertical towards west.

S97 That is clear. Just one aside. We have been using the word 'particle' quite often. I should have asked this before. Is a particle the same thing as a body?

T97 A particle is an idealization. You might consider it a body of negligible, ideally zero, size.

Box 3.2

Let S be a certain frame of reference and S' be another one moving with uniform velocity u with respect to S . (See Fig.3.9.) R is the position vector of O' (the origin of S' frame) with respect

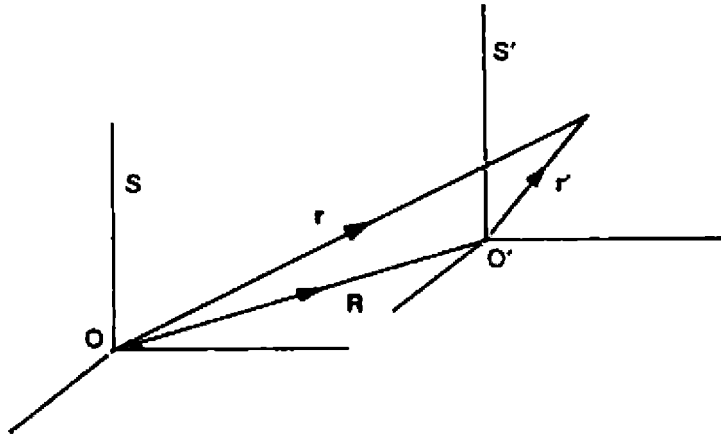


Fig.3.9 Two frames S and S' in relative motion.

to the S frame. r and r' denote the position vectors of a particle with respect to the S and the S' frames respectively. It is clear from the figure that

$$r = r' + R.$$

We differentiate this with respect to time and identify $dR/dt = u$, $dr/dt = v$ and $dr'/dt = v'$ to obtain the velocity addition rule

$$v = v' + u.$$

S98 *O. K. but what is the point in this idealization ?*

T98 You see, we are focussing here on one type of motion, the so-called translational motion. We are ignoring all other modes of motions that a body might have : rotation, vibration etc. It is, therefore, best to consider an idealized body which does not have these latter modes of motion but has only translational motion. A point object, i.e.

a particle, cannot possess any other type of motion except translation.[†]

S99 *What is the use of considering an idealized body that does not exist, and ignoring realistic aspects of motion ?*

T99 This is a rather general question. It really comes down to the basic strategy of doing physics, or at least learning physics. A realistic problem such as the general motion of an arbitrary body is complicated by many features. If you try to address all the aspects right from the beginning, life will be complicated and chances are that you will not get anywhere. It is a good practice, therefore, to truncate the problem; make simplifying idealizations which enable one to focus one's attention on limited aspects of the problem. Later, when you have mastered various aspects, you can put them all together and, hopefully, tackle the general problem. It is not a question of ignoring anything. It is a matter of strategy for handling a complex situation step by step. The process of idealization is basic to this strategy.

Kinematics versus dynamics

S100 *One last question. We have used the word 'kinematics' in the title of this chapter. What does one mean by the term ?*

T100 Kinematics deals with quantities which characterize motion : displacement, velocity, acceleration etc.; the relations between them, and their transformation properties from one frame to another - things which follow from the very definitions of these quantities. We know that accelerations are caused by forces but investigation of these causes is not the concern here. When we examine the causes of accelerations in a problem we enter the domain of dynamics.

S101 *I have heard something like that before, but can you explain the point through some examples ?*

[†] Note, the oscillatory motion of a point particle as also its rotational motion are instances of its translational motion. It is wrong to associate these motions with vibrational and rotational degrees of freedom. The latter can arise only for an extended body.

T101 The kinematic relations in T35 or more generally in T54, for example, follow from the definitions of velocity and acceleration. They have nothing to do with the laws of motion which relate force and acceleration. If these laws were different, the kinematic relations would nonetheless be the same. For instance, the laws of motion for inertial observers are different from those for non-inertial observers. But given acceleration in a frame, the same kinematic relations hold good whatever the frame may be.

As another example, the Galilean rule of velocity addition that we used in T96 is a kinematic relation, not a dynamical law.

Yet another example of kinematic relations is the Lorentz transformation equations of special relativity. These relate positions and times in one frame to those in another. They follow directly from the operational definitions of these quantities and the postulate of the constancy of velocity of light.

On the other hand the relation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \quad \text{where} \quad \mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

is a dynamical law of relativistic mechanics. In the same way Maxwell's equations are dynamical laws of electromagnetism. So is the Lorentz force equation for a particle of charge q in an electromagnetic field,

$$\frac{d\mathbf{p}}{dt} = q (\mathbf{E} + \mathbf{v} \times \mathbf{B}) .$$

To sum up, kinematics involves merely definitions of quantities characterizing motion and the relations that follow from these definitions. Dynamics involves knowing the nature of forces and interactions and how they cause motion. Dynamics, as they say, is real physics.

S102 *I am waiting to plunge into it.*